Journal of Global Positioning Systems (2009)
Vol.8, No.1 :86-96

Journal of Global Positioning Systems
2009
© CPGPS

Reliability Analysis in Kalman filtering

Jian-Guo Wang
Faculty of Science and Engineering, York University, jgwang@yorku.ca

Abstract

This manuscript centers on the reliability theory and its applications in Kalman filtering. Especially, it delivers a distinct derivation of the redundancy contribution - the key element of reliability theory for the Kalman filter algorithm that has not been comprehensively discussed in literature at present. A distinction is made between the system innovation vector and the measurement (or pseudo measurement) residual vector. This allows to directly analyse the observation vector and the process noise vector. Particular attention is paid not only to the theoretical fundamentals of the reliability, and also to the introduction of some practical applications about the use of the redundancy contribution in Kalman filtering. The manuscript aims at assisting readers in a comprehensive understanding of reliability analysis in Kalman filtering.

Keywords: Kalman filter, stochastic information, quality control, internal reliability, external reliability, redundancy contribution, controllable value, minimal detectable outlier.

---

1 Introduction

Quality control is a system of maintaining standards in manufactured products by testing and inspection [Barber, 1998]. It belongs to one of the paramount tasks to the applications with Kalman filter just as its importance to the traditional geodetic applications. Specifically, the term quality here comprises of reliability and precision [Salzmann and Teunissen, 1989]. The former describes the ability of the redundant observations to check model errors, or is concerned with the effects of possible misspecifications of the model on the estimation results, whilst the latter measures the spread of the estimation results due to the stochastic model and is represented by a covariance matrix.

The measures for quality control have been affirmatively developed together with the development of the theory of Kalman filter and mostly driven by specific applications [Mehra & Peschon, 1971; Willsky et al, 1974, 1975; Willsky, 1976; Salzmann & Teunissen, 1989; Lu, 1991; Salzmann, 1991, 1993; Gao, 1992; Wang, 1997, 2008; Caspary & Wang, 1998; Tiberius, 1998; Hewitson, 2006; Wang et al, 2009; etc.]. As compared with the reliability analysis, the analysis of precision or accuracy has greatly matured in the years.

The reliability theory in the method of least squares was adapted for quality control in Kalman filtering by [Salzmann and Teunissen, 1989], in which the testing procedure, reliability, estimation of variance components and certain practical considerations were outlined. [Teunissen, 1990; Salzmann, 1993] further suggested the DIA (Detection, Identification and Adaptation) procedure for quality control in integrated navigation system that has been often quoted in literature [Lu, 1991; Gao, 1992; Wang, 1996, 2008; Hewitson, 2006; etc]. The reliability analysis, statistic tests and variance component estimation were mostly derived either from the system innovation of Kalman filter [Mehra, 1971; Willsky, 1976; Teunissen, 1990; etc.] or from the latest available measurement vector and the predicted state vector blended with the process noise as the pseudo-measurement vector [Gao, 1992; Salzmann, 1993; Jia, et al, 1998; Tiberius, 1998; Hewitson, 2006; etc.]. An alternate derivation of reliability measures in Kalman filtering was given by [Wang, 1997], in which the most novel gain was the clear expression of the redundancy contribution of measurements in Kalman filtering for the latest available measurement vector, the process noise vector, and the predicted state vector without having blended with the process noise, respectively. As it has been in least squares adjustment, the redundancy contribution is essential to statistic tests, variance component estimation, and the reliability analysis overall in Kalman filtering [Wang, 1997, 2008; Caspary & Wang, 1998; Wang et al, 2009].

The author attempts to systematically provide a manuscript about the reliability theory in Kalman filtering in a practical way so that readers can have a systematic and comprehensive grasp of the subject. The reliability concept is summarized in Section 2. The core of this text lies in Section 3, which first defines the standard model for Kalman filter, and then describes an alternate deviation of the Kalman filter algorithm, and ends with the delivery of the redundancy contribution for Kalman filter. Section 4 gives a numerical example about the reliability analysis in Kalman filtering based on a simulated 2D land vehicle trajectory. Some of the useful applications based on the redundancy contribution in Kalman filtering are given in Section 5. This manuscript is ended with the concluding remarks in Section 6.

2. CONCEPT OF RELIABILITY IN LEAST SQUARES ESTIMATION

The reliability theory was initially founded in the method of least squares by [Baarda, 1968]. For more specific details about the reliability analysis refer to [Jäger & Bill, 1986; Caspary, 1988; Li & Yuan, 2002; Leick, 2004; etc]. Measures of accuracy and measures of reliability form together a sufficient basis for the assessment and comparison of the quality of geodetic networks [Caspary, 1988]. This is also true for the same purpose in Kalman filtering.

Before this section goes into details, a necessary clarification must be made between the measures of reliability of least squares estimation and the ones of reliability of manufacturing processes and of products. The reliability measures used in quality control for the latter are usually functions of the time of continuous proper functioning of a device or a part of thereof [Caspary, 1988]. For example, the most commonly used reliability measure for military equipment is the so-called MTBF (Mean Time Between Failures). The reliability measures that are being discussed for the method of least squares can be very different from the ones used in industrial process. However, the fundamental ideas in quality control have stimulated geodetic scientists to develop a concept of reliability for the method of least squares (or Gaussian Markov Model) as in [Baarda, 1968].

Reliability in least squares estimation refers to the controllability of measurements, i.e., the ability to detect outliers and to estimate the effects that the undetected outliers may have on the system solution [Leick, 2004]. Therefore, the criteria of reliability in least squares estimation can be classified as [Caspary, 1988; etc.]:

- The internal reliability and
- The external reliability.

which will be summarized in the following subsections.

2.1. Statistical basics

As is well known, a very prime and straightforward statistical rule, 3σ, has commonly been employed to identify the measurement outliers. In reality, the outliers on measurements may not be able to be effectively removed in this way. Otherwise, it would not be necessary to introduce the reliability concept into the theory of least squares estimation. The best way to learn about this fact is to study the relation between the measurement errors and their residuals.

Without loss of generality, a linearized observation equation system is considered

$$\boldsymbol{L}+\boldsymbol{v}=\boldsymbol{B}\delta\boldsymbol{x}+\boldsymbol{F}(\boldsymbol{x}^{(0)}) \tag{1}$$

where $\boldsymbol{L}$ is the $\boldsymbol{n}\times 1$ observation vector; $\boldsymbol{v}$ is the $\boldsymbol{n}\times 1$ residual vector of $\boldsymbol{L}$; $\boldsymbol{x}$ is the $\boldsymbol{t}\times 1$ parameter vector with a vector $\boldsymbol{x}^{(0)}$ of known approximate values and the correction vector $\boldsymbol{\delta\hat{x}}$ for $\boldsymbol{x}^{(0)}$; $\boldsymbol{F}(\boldsymbol{x})$ is the $\boldsymbol{n}\times 1$ vector as nonlinear mathematical function of $\boldsymbol{x}$ for $\boldsymbol{L}$; $\boldsymbol{B}$ is the $\boldsymbol{n}\times\boldsymbol{t}$ design matrix that is composed of the partial derivatives of $\boldsymbol{F}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}^{(0)}$. The observation vector $\boldsymbol{L}$ is normally distributed as $\boldsymbol{L}\sim\boldsymbol{N}(\tilde{\boldsymbol{L}},\boldsymbol{D}_{\boldsymbol{ll}})$ with its expectation vector $\tilde{\boldsymbol{L}}$ and its variance matrix $\boldsymbol{D}_{\boldsymbol{ll}}$. In practice, $\boldsymbol{D}_{\boldsymbol{ll}}$ is given as

$$\boldsymbol{D}_{\boldsymbol{LL}}=\boldsymbol{\sigma}_0^2\boldsymbol{P}_{\boldsymbol{ll}}^{-1}=\boldsymbol{\sigma}_0^2\boldsymbol{Q}_{\boldsymbol{ll}} \tag{2}$$

where σ_0^2 is the variance of unit weight and $\boldsymbol{P}_{\boldsymbol{ll}}^{-1}$ and $\boldsymbol{Q}_{\boldsymbol{ll}}$ are the weight matrix and the cofactor matrix of $\boldsymbol{L}$, respectively. $\boldsymbol{Q}_{\boldsymbol{ll}}$ is also called as the inverse of a weight matrix and interchangeably used together with the weight matrix.

The least-squares (LS) solution of (1) is

$$\hat{\boldsymbol{x}}=\boldsymbol{x}^{(0)}+\boldsymbol{\delta}\hat{\boldsymbol{x}}=\boldsymbol{x}^{(0)}+\boldsymbol{N}^{-1}\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{P}_{\boldsymbol{ll}}(\boldsymbol{L}-\boldsymbol{F}(\boldsymbol{x}^{(0)})) \tag{3}$$

with its variance matrix

$$\boldsymbol{D}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}=\hat{\boldsymbol{\sigma}}_0^2(\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{P}_{\boldsymbol{ll}}\boldsymbol{B})^{-1} \tag{4}$$

Where

$$\hat{\boldsymbol{\sigma}}_0^2=\frac{\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{P}_{\boldsymbol{ll}}\boldsymbol{v}}{\boldsymbol{r}} \tag{5}$$

The degree of freedom $\boldsymbol{r}$ in (5) is quantitatively equal to the number of the redundant measurements of (1), $\boldsymbol{n}-\boldsymbol{t}$ $(\boldsymbol{n}>\boldsymbol{t})$. The measurement residual vector is given by

$$\boldsymbol{v}=-\boldsymbol{Q}_{\boldsymbol{vv}}\boldsymbol{Q}_{\boldsymbol{ll}}^{-1}(\boldsymbol{L}-\boldsymbol{F}(\boldsymbol{x}^{(0)})) \tag{6}$$

with its cofactor matrix

$$\boldsymbol{Q}_{\boldsymbol{vv}}=\boldsymbol{Q}_{\boldsymbol{ll}}-\boldsymbol{B}(\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{Q}_{\boldsymbol{ll}}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\boldsymbol{T}} \tag{7}$$

If the true parameter vector $\tilde{\boldsymbol{x}}$ is provided instead of $\boldsymbol{x}^{(0)}$, the vector $\boldsymbol{L}-\boldsymbol{F}(\boldsymbol{x}^{(0)})$ only consists of the true measurement error as ε so that (6) becomes to

$$\boldsymbol{v} = -\boldsymbol{Q}_{vv}\boldsymbol{Q}_{ll}^{-1}\varepsilon \quad (8)$$

Due to the fact that $\boldsymbol{Q}_{vv}\boldsymbol{Q}_{ll}^{-1}$ is an idempotent matrix, its trace is equal to the number of the redundant measurements of (1). Its individual diagonal elements, denoted as $\boldsymbol{r}_i$ ($\boldsymbol{i} = 1,2,...,\boldsymbol{n}$), characterize the distribution of the redundancy of the system and are called as redundancy contribution or the redundancy indices of the measurements. If $\boldsymbol{Q}_{ll}$ is diagonal, which means the measurements in $\boldsymbol{L}$ being not correlated, the redundancy index of a measurement lies within (0, 1).

A thorough analysis of (8) can show (i) how the measurement residuals are affected by each of the measurement errors, (ii) how the individual measurement residuals are affected by the error on a specific measurement, and (iii) how the error on a specific measurement affects its own residual [Hahn & Mierlo, 1986; Li & Yuan, 2002; etc.].

Based on the measurement residual vector, the well-known data snooping was constructed as follows [Baarda, 1968]:

$$\boldsymbol{w}_i = \frac{\boldsymbol{v}_i}{\sigma_{v_i}} = \frac{\boldsymbol{v}_i}{\sqrt{\boldsymbol{r}_i}\sigma_{L_i}} \quad (9)$$

If the $\boldsymbol{i}$–th measurement contains an outlier $\nabla\boldsymbol{L}_i$, it affects its own measurement residual by the magnitude of

$$\nabla\boldsymbol{v}_i = -\boldsymbol{r}_i\nabla\boldsymbol{L}_i \quad (10)$$

and results in a noncentrality parameter of $\boldsymbol{w}_i$

$$\delta_i = \nabla\boldsymbol{w}_i = -\frac{\sqrt{\boldsymbol{r}_i}}{\sigma_{L_i}}\nabla\boldsymbol{L}_i \quad (11)$$

Against a potential measurement outlier, the null hypothesis based on (9) is given by

$$\boldsymbol{H}_0 : \boldsymbol{w}_i = \frac{\boldsymbol{v}_i}{\boldsymbol{\sigma}_{v_i}} \sim \boldsymbol{N}(0,1) \quad (12)$$

versus the alternative hypothesis based on (11)

$$\boldsymbol{H}_a : \boldsymbol{w}_i = \frac{\boldsymbol{v}_i}{\boldsymbol{\sigma}_{v_i}} \sim \boldsymbol{N}(\boldsymbol{\delta}_i,1) \quad (13)$$

A statistic test is always accompanied by the probability errors (Type I error and Type II error) with respect to the significance level and the power of a test (Table 1). These two types of errors cannot be minimized at the same time. The bigger Type I error one speifies, the higher test power one can gain. Studying how to balance these two types of errors is a good transition point to the subject of reliability.

Table 1: Hypothesis Test

Decision	$\boldsymbol{H}_0$ true	$\boldsymbol{H}_0$ false
Accept $\boldsymbol{H}_0$	correct	Type II error
Rejct $\boldsymbol{H}_0$	Type I error	correct

2.2. **The Internal Reliability**

The internal reliability refers to the desired model property of facilitating the detection of sytematic errors and the localization of outliers without requiring addtional information (self-checking model) [Caspary, 1988]. It is a measure of the capability of the system (1) to detect measurement outliers with the given probability. The analysis of the internal reliability of a system can be performed based on the given system structure without having the real measurements or their residuals available. The overview here will focus on three commonly used measures of the internal reliability.

First, how a given error $\nabla\boldsymbol{L}_i$ affects the residual $\nabla\boldsymbol{v}_i$ is controlled by the redundancy contribution $\boldsymbol{r}_i$ of the same measurement according to (10). Hence, the redundancy contribution can be directly used as a measure of internal reliability. Obviously, a system designer expects to have an evenly distributed redundancy among all of the observations in general. By computing the values of $\boldsymbol{r}_i$ for all measurements, one can assess the internal reliability of the system because the individual redundancy indices can show the weak parts of the system and advice the necessary improvement accordingly in term of reliability. Apparently, the redundancy contribution can be either used as a global, or a local measure of the internal reliability.

Secondly, what is the minimal detectable outlier in order to be able to identify an outlier on an observation at a significance level of $\boldsymbol{\alpha}_0$ and with the test power of $1-\beta_0$? Based on the normal distribution, one can determine the distance $\boldsymbol{\delta}_i$ between $\boldsymbol{H}_0$ and $\boldsymbol{H}_a$

$$\delta_0 = \nabla_0\boldsymbol{w}_i = \delta_0(\alpha_0,\beta_0) \quad (14)$$

so that the value of the minimal detectable outlier of a measurement is estimated after (11):

$$\nabla_0 \boldsymbol{L_i} = \sigma_{L_i} \frac{\delta_0}{\sqrt{\boldsymbol{r_i}}} \tag{15}$$

which is also called as the critical value for the internal reliability. Therefore, another measure of internal reliability of a measurement can be defined as the controllable value

$$\boldsymbol{c}_{0\boldsymbol{i}} = \frac{\boldsymbol{\delta}_0}{\sqrt{\boldsymbol{r_i}}} \tag{16}$$

that tells how many times of the measurement standard deviation the minimal detectable outlier on a specific measurement is equal to. $\boldsymbol{c}_{0\boldsymbol{i}}$ is a unitless number and only depends on the geometry of the system reflected by $\boldsymbol{r_i}$ and the given $\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_0$. This measure may be categorized as a local measure of internal reliability.

Thirdly, for the quadratic form of the residual vector $\boldsymbol{v}$ in (6), the global model test runs

$$\frac{\boldsymbol{v^T P_{ll} v}}{\boldsymbol{\sigma}_0^2} \sim \boldsymbol{\chi}^2(\boldsymbol{r}, \boldsymbol{\lambda}) \tag{17}$$

based on the a priori variance of weight unit $\boldsymbol{\sigma}_0^2$ with its degree of freedom $\boldsymbol{r}$ and a noncentrality parameter $\boldsymbol{\lambda}$. This noncentrality parameter is determined as

$$\boldsymbol{\lambda} = \boldsymbol{e^T P_{ll} Q_{vv} P_{ll} e} / \boldsymbol{\sigma}_0^2 \tag{18}$$

under the assumption that an error in the functional model (1) can be compensated by a vector $\boldsymbol{e}$ as follows

$$\boldsymbol{L} = \boldsymbol{B}\delta\boldsymbol{x} - \boldsymbol{e} + \boldsymbol{F}(\boldsymbol{x}^{(0)}) + \varepsilon \tag{19}$$

On the ground of the Rayleigh inequality, the upper limit of the noncentrality parameter can be given by

$$\boldsymbol{\lambda} = \boldsymbol{e^T P_{ll} Q_{vv} P_{ll} e} / \boldsymbol{\sigma}_0^2 \le \boldsymbol{\lambda}_{\max} \boldsymbol{e^T e} / \boldsymbol{\sigma}_0^2 \tag{20}$$

where $\boldsymbol{\lambda}_{\max}$ is the maximum eigenvalue of $\boldsymbol{P_{ll} Q_{vv} P_{ll}}$. Hence, $\boldsymbol{\lambda}_{\max}$ may be utilized as a global measure of internal reliability, although its usefulness must not be overrated [Caspary, 1988].

2.3. **The External Reliability**

The external reliability measures the response of the model to undetected model errors such as systematic errors and measurement outliers [Caspary, 1988]. It studies the impact of undetectable model errors on the estimated parameters or on a specific function of the parameters. It is essential for a system designer to gain full knowledge of the effect of the potential model errors on the estimated parameters because one cannot generally expect to have a perfect systematic error and measurement outlier detection in the data processing. In term of external reliability, a high quality of the system will insignificantly to undetected or unmodeled errors.

By denoting the bias vector of the parameter vector as $\Delta\boldsymbol{x}$, it can mathematically be connected to $\boldsymbol{e}$ in (19)

$$\Delta\hat{\boldsymbol{x}} = (\boldsymbol{B^T P_{ll} B})^{-1} \boldsymbol{B^T P_{ll} e} \tag{21}$$

wherein $\boldsymbol{e}$ is an error vector either having non-zero components on all of the measurements, on some of them, or even only on one single measurement. As a matter of course, it provides the logical way to connect the external reliability analysis with the internal reliability analysis by first analysing the internal reliability through the minimal detectable values of errors as in (15), and then further studying their effect on the parameters or on a function of them.

As a common practice, the average measure of the effect of $\boldsymbol{e}$ is the quadratic form of $\Delta\boldsymbol{x}$

$$\Delta\hat{\boldsymbol{x}}^{\boldsymbol{T}} \boldsymbol{Q}_{xx} \Delta\hat{\boldsymbol{x}} = \boldsymbol{e^T P_{ll} B Q_{xx} B^T P_{ll} e} \tag{22}$$

Analog to (20), the Rayleigh inequality tells

$$\boldsymbol{e^T P_{LL} B Q_{xx} B^T P_{LL} e} \le \lambda'_{\max} \boldsymbol{e^T e} \tag{23}$$

where $\lambda'_{\max}$ is the maximum eigenvalue of $\boldsymbol{P_{ll} B Q_{xx} B^T P_{ll}}$. $\lambda'_{\max}$ can be used as a measure of global external reliability of the system [Caspary, 1988]. The similar measure of the external reliability for a function of the parameters is easily established on the analog of (22) and (23).

A measure of local external reliability can be established through studying how the parameter vector is affected by an error on one single measurement:

$$\delta_{0,\boldsymbol{i}} = \Delta\hat{\boldsymbol{x}}^{\boldsymbol{T}}(\boldsymbol{i}) \boldsymbol{Q}_{xx} \Delta\hat{\boldsymbol{x}}(\boldsymbol{i}) = \boldsymbol{e}_{\boldsymbol{i}}^2 \boldsymbol{p}_{\boldsymbol{i}}^2 \boldsymbol{b}_{\boldsymbol{i}}^{\boldsymbol{T}} \boldsymbol{Q}_{xx} \boldsymbol{b}_{\boldsymbol{i}} \tag{24}$$

where $\boldsymbol{e_i}$ is an error on the $\boldsymbol{i}$-th measurement, $\boldsymbol{p_i}$ is the weight of the same measurement, $\boldsymbol{b_i}$ is the $\boldsymbol{i}$-th column vector of the design matrix $\boldsymbol{B}$. By substituting δ_0 in (14) into (22), (24) can be simplified to [Li & Yuan, 2002]

$$\delta_{0,\boldsymbol{i}} = \delta_0 \sqrt{\frac{1 - \boldsymbol{r_i}}{\boldsymbol{r_i}}} \qquad (\boldsymbol{i} = 1, 2, ..., \boldsymbol{n}) \tag{25}$$

which is one of the most commonly used measures for the external reliability.

3. RELIABILITY IN KALMAN FILTERING

This section is to provide first a brief overview of the algorithm of the discrete Kalman filter, secondly an alternate formulation of it in order to be able to adapt the reliability theory of the least squares estimation described in Section 2 to the algorithm of Kalman filter, and thirdly the distinct expression of the redundancy contribution of individual measurements in Kalman filtering. Unquestionably, a comprehensive reliability analysis can be applied to practical applications based on the derived redundancy contribution for Kalman filter.

3.1. The Kalman filtering algorithm

As usual, a linear or linearized system with the state-space notation is considered under the assumption that the data are available over a discrete time series $\{t_0, t_1, \ldots, t_N\}$, which will often be simplified to $\{0,1,\ldots,N\}$. Without loss of generality, a deterministic system input vector will be droped in all of the expressions in this paper. Hence, at any time instant k ($1 \le k \le N$) the system can be written as follows:

$$x(k) = A(k-1)x(k-1) + B(k-1)w(k-1) \tag{26}$$

$$z(k) = C(k)x(k) + \Delta(k) \tag{27}$$

where $x(k)$ is the $n \times 1$ state-vector, $z(k)$ is the $p \times 1$ observation vector, $w(k-1)$ is the $m \times 1$ process noise vector, $\Delta(k)$ is the $p \times 1$ measurement noise vector, $A(k-1)$ is the $n \times n$ coefficient matrix of $x(k)$, $B(k-1)$ is the $n \times m$ coefficient matrix of $w(k-1)$, $C(k)$ is the $p \times n$ coefficient matrix of $z(k)$. The random vectors $w(k)$ and $\Delta(k)$ are generally assumed to be Gaussian with zero-mean:

$$w(k) \sim N(o, Q(k)) \tag{28}$$

$$\Delta(k) \sim N(o, R(k)) \tag{29}$$

where $Q(k)$ and $R(k)$ are positive definite variance matrices, respectively. Further assumptions about the random noise are made and specified as follows ($i \ne j$):

$$Cov(w(i), w(j)) = O \tag{30}$$

$$Cov(\Delta(i), \Delta(j)) = O \tag{31}$$

$$Cov(w(i), \Delta(j)) = O \tag{32}$$

Very often, we also have to assume the initial mean and variance-covariance matrix $x(0)$ and $D_{xx}(0)$ for the system state. In addition, the initial state $x(0)$ is also assumed to be independent of $w(k)$ and $\Delta(k)$ for all k.

The optimal estimate $\hat{x}(k)$ of $x(k)$ can be derived by applying, for example, the least-squares method in the sense of unbiasedness and minimum variance. Table 2 summarizes its solution only for the need of further development of this manuscript:

An essential characteristic of the sequence $d(1)$, …, $d(i)$, …, $d(k)$ is that they are independent from each other epochwise [Stöhr, 1986; Chui, Chen, 2009]:

$$Cov\{d(i), d(j)\} = O \quad \text{for } (i \ne j) \tag{33}$$

Table 2: Solution of Kalman filter

The predicted state vector and its variance matrix $\hat{x}(k/k-1) = A(k-1)\hat{x}(k-1)$ $D_{xx}(k/k-1) = A(k-1)D_{xx}(k-1)A^T(k-1) + B(k-1)Q(k-1)B^T(k-1)$
The optimal estimated state vector and its variance matrix $\hat{x}(k) = \hat{x}(k/k-1) + G(k)d(k)$ $D_{xx}(k) = G(k)R(k)G^T(k) + [E - G(k)C(k)]D_{xx}(k/k-1)[E - G(k)C(k)]$
The optimal estimated state vector and its variance matrix $d(k) = z(k) - C(k)\hat{x}(k/k-1)$ $D_{dd}(k) = C(k)D_{xx}(k/k-1)C^T(k) + R(k)$
The Kalman gain matrix $G(k) = D_{xx}(k/k-1)C^T(k)D_{dd}^{-1}(k)$

3.2. An alternate formulation of Kalman filter

The optimal estimate $\hat{x}(k)$ of $x(k)$ at the instant k is always associated with the stochastic information, which may be divided into three independent groups:

a. The real observation noise $\Delta(k)$,
b. The system noise $w(k-1)$,
c. The noise from the predicted $\hat{x}(k/k-1)$ through $\hat{x}(k-1)$, on which $\{\Delta(1), \ldots, \Delta(k-1)\}$ and $\{w(0), w(1), \ldots, w(k-2)\}$ are propagated into the current system state model.

Traditionally, either the system innovation sequences, or the synthesized prediction of $\hat{x}(k/k-1)$ from "b" and "c" has been statistically analysed. If these different error resources could be studied separately, it could be very helpful to the performance evaluation of (1). For this purpose, the model in 3.1 can be reconstituted using three groups of residual equations as follows [Wang, 1997; Caspary & Wang 1998; Wang, 2008, 2009]:

$$\left.\begin{aligned} \boldsymbol{v}_{\boldsymbol{l}_x}(\boldsymbol{k}) &= \quad \hat{\boldsymbol{x}}(\boldsymbol{k}) - \boldsymbol{B}(\boldsymbol{k}-1)\ \hat{\boldsymbol{w}}(\boldsymbol{k}-1) \quad \boldsymbol{l}_x(\boldsymbol{k}) \\ \boldsymbol{v}_{\boldsymbol{l}_w}(\boldsymbol{k}) &= \quad \hat{\boldsymbol{w}}(\boldsymbol{k}-1) - - \boldsymbol{l}_w(\boldsymbol{k}) \\ \boldsymbol{v}_{\boldsymbol{l}_z}(\boldsymbol{k}) &= \boldsymbol{C}(\boldsymbol{k})\ \hat{\boldsymbol{x}}(\boldsymbol{k}) \quad - \boldsymbol{l}_z(\boldsymbol{k}) \end{aligned}\right\} \quad (34)$$

which are corresponding to the following independent (pseudo-)observation groups

$$\left.\begin{aligned} \boldsymbol{l}_x(\boldsymbol{k}) &= \boldsymbol{A}(\boldsymbol{k}-1)\hat{\boldsymbol{x}}(\boldsymbol{k}-1) \\ \boldsymbol{l}_w(\boldsymbol{k}) &= \boldsymbol{w}_0(\boldsymbol{k}-1) \\ \boldsymbol{l}_z(\boldsymbol{k}) &= \boldsymbol{z}(\boldsymbol{k}) \end{aligned}\right\} \quad (35)$$

along with their variance-covariance matrices:

$$\left.\begin{aligned} \boldsymbol{D}_{\boldsymbol{l}_x\boldsymbol{l}_x}(\boldsymbol{k}) &= \boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k}-1)\boldsymbol{A}^T(\boldsymbol{k}-1) \\ \boldsymbol{D}_{\boldsymbol{l}_w\boldsymbol{l}_w}(\boldsymbol{k}) &= \boldsymbol{Q}(\boldsymbol{k}-1) \\ \boldsymbol{D}_{\boldsymbol{l}_z\boldsymbol{l}_z}(\boldsymbol{k}) &= \boldsymbol{R}(\boldsymbol{k}) \end{aligned}\right\} \quad (36)$$

Here, $\boldsymbol{l}_x(\boldsymbol{k})$, $\boldsymbol{l}_w(\boldsymbol{k})$ and $\boldsymbol{l}_z(\boldsymbol{k})$ are the $\boldsymbol{n}\times 1$, $\boldsymbol{m}\times 1$ and $\boldsymbol{p}\times 1$ measurement or pseudo-measurement vectors, respectively. Usually $\boldsymbol{w}_0(\boldsymbol{k}-1) = \boldsymbol{o}$.

The identical estimate $\hat{\boldsymbol{x}}(\boldsymbol{k})$ of $\boldsymbol{x}(\boldsymbol{k})$ as in 3.1 can be obtained by applying the least squares principle to (34)-(36) [Wang, 1997].

This novel formulation of Kalman filter algorithm has directly made the measurement residual vectors available for error analysis. One can definitely take its advantages to build up the test statistics in Kalman filter based on the measurement residual vectors [Wang, 2008]. In this way, any of three measurement vectors can be separately analysed through their own residual vectors.

3.3. The redundancy contribution in Kalman filtering

This subsection delivers the key element of the reliability theory for Kalman filter – the redundancy contribution of measurements. It is understood that the diagonal elements of the idempotent matrix $\boldsymbol{Q}_{vv}\boldsymbol{Q}_{ll}^{-1}$, as discussed in Section 2.1, represent the redundancy contribution of measurements in the method of least squares, so does in Kalman filtering.

Straightforward without the superfluous in-between steps, the measurement residual vectors are given as the functions of the system innovation vector at each epoch

$$\boldsymbol{v}_{\boldsymbol{l}_x\boldsymbol{l}_x}(\boldsymbol{k}) = \boldsymbol{D}_{\boldsymbol{l}_x\boldsymbol{l}_x}(\boldsymbol{k})\boldsymbol{D}_{xx}^{-1}(\boldsymbol{k}/\boldsymbol{k}-1)\boldsymbol{K}(\boldsymbol{k})\boldsymbol{d}(\boldsymbol{k}) \quad (37)$$

$$\boldsymbol{v}_{\boldsymbol{l}_w\boldsymbol{l}_w}(\boldsymbol{k}) = \boldsymbol{Q}(\boldsymbol{k}-1)\boldsymbol{B}^T(\boldsymbol{k}-1)\boldsymbol{D}_{xx}^{-1}(\boldsymbol{k}/\boldsymbol{k}-1)\boldsymbol{K}(\boldsymbol{k})\boldsymbol{d}(\boldsymbol{k}) \quad (38)$$

$$\boldsymbol{v}_{\boldsymbol{l}_z\boldsymbol{l}_z}(\boldsymbol{k}) = \{\boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k}) - \boldsymbol{E}\}\boldsymbol{d}(\boldsymbol{k}) \quad (39)$$

with their variance matrices:

$$\boldsymbol{D}_{\boldsymbol{v}_{\boldsymbol{l}_x}\boldsymbol{v}_{\boldsymbol{l}_x}}(\boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k}-1)\boldsymbol{A}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k}) \rightarrow \\ \boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})\boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k}-1)\boldsymbol{A}^T(\boldsymbol{k}-1) \quad (40)$$

$$\boldsymbol{D}_{\boldsymbol{v}_{\boldsymbol{l}_w}\boldsymbol{v}_{\boldsymbol{l}_w}}(\boldsymbol{k}) = \boldsymbol{Q}(\boldsymbol{k}-1)\boldsymbol{B}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k}) \rightarrow \\ \boldsymbol{C}(\boldsymbol{k})\boldsymbol{B}(\boldsymbol{k}-1)\boldsymbol{Q}(\boldsymbol{k}-1) \quad (41)$$

$$\boldsymbol{D}_{\boldsymbol{v}_{\boldsymbol{l}_z}\boldsymbol{v}_{\boldsymbol{l}_z}}(\boldsymbol{k}) = \{\boldsymbol{E} - \boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k})\}\boldsymbol{R}(\boldsymbol{k}) \quad (42)$$

after one substitutes (36) for $\boldsymbol{D}_{\boldsymbol{l}_x\boldsymbol{l}_x}(\boldsymbol{k})$, $\boldsymbol{D}_{\boldsymbol{l}_w\boldsymbol{l}_w}(\boldsymbol{k})$ and $\boldsymbol{D}_{\boldsymbol{l}_z\boldsymbol{l}_z}(\boldsymbol{k})$. Similar to (33), we can readily prove the following results of independence:

$$\boldsymbol{Cov}\{\boldsymbol{v}(\boldsymbol{i}), \boldsymbol{v}(\boldsymbol{j})\} = \boldsymbol{O} \quad \text{for } (\boldsymbol{i} \neq \boldsymbol{j}) \quad (43)$$

Under the assumption that the observations in $\boldsymbol{l}_z(\boldsymbol{k})$, i.e. $\boldsymbol{z}(\boldsymbol{k})$, are uncorrelated, $\boldsymbol{R}(\boldsymbol{k})$ becomes diagonal. In this case, the redundancy index of each component in $\boldsymbol{z}(\boldsymbol{k})$ is given by

$$\boldsymbol{r}_{z_i}(\boldsymbol{k}) = 1 - \{\boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k})\}_{ii} \quad (44)$$

In a similar way, if the noise factors in $\boldsymbol{l}_w(\boldsymbol{k})$, i.e. $\boldsymbol{w}(\boldsymbol{k}-1)$, are not correlated, $\boldsymbol{Q}(\boldsymbol{k}-1)$ becomes diagonal. The redundancy index of each process noise factor is equal to

$$\boldsymbol{r}_{w_i}(\boldsymbol{k}) = [\boldsymbol{Q}(\boldsymbol{k}-1)\boldsymbol{B}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})\boldsymbol{B}(\boldsymbol{k}-1)]_{ii} \quad (45)$$

However, the pseudo-observations in $\boldsymbol{l}_x(\boldsymbol{k})$, the predicted state vector, are generally correlated. Therefore, the total redundancy contribution of $\boldsymbol{l}_x(\boldsymbol{k})$ cannot be decomposed to its individual components. More about the reliability measures for the correlated measurements refers to [Li & Yuan, 2002; Chen & Wang, 1996]

Furthermore, three independent measurements groups as in (35) make the following redundancy contribution:

$$\boldsymbol{r}_x(\boldsymbol{k}) = \\ \boldsymbol{trace}\{\boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k})\boldsymbol{A}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})\} \quad (46)$$

$$r_w(k) = trace\{Q(k-1)B^T(k-1)C^T(k)D_{dd}^{-1}(k)C(k)B(k-1)\} \quad (47)$$

$$r_z(k) = trace[E - C(k)K(k)] \quad (48)$$

to the entire system, respectively. It is satisfied with:

$$r(k) = r_x(k) + r_w(k) + r_z(k) = p(k) \quad (49)$$

where $p(k)$ is the number of the total redundant measurements at epoch k and is logically equal to the dimension of the real measurement vector $z(k)$.

Based on the redundancy contribution provided above, the reliability analysis in Kalman filtering can be introduced exactly in the same way as in least squares method. Sufficient discussion may be found in [Wang, 1997].

4. Numerical example

This section provides a numerical example about the redundancy contribution together with the internal and external reliability in Kalman filtering. A 2D land vehicle trajectory is simulated based on the simplified uniform circular movement [Wang, 1997]. The similar case studies can also be found in [Ramm, 2008; Eichhorn, 2005].

In the 2D mapping frame (Fig. 1), the kinematic system at the instant $k+1$ is expressed as follows

$$\left.\begin{aligned} x(k+1) &= x(k) + v_t(k)\cos\varphi(k)(t_{k+1} - t_k) \\ y(k+1) &= y(k) + v_t(k)\sin\varphi(k)(t_{k+1} - t_k) \\ \varphi(k+1) &= \varphi(k) \\ v_t(k+1) &= v_t(k) \end{aligned}\right\} \quad (50)$$

with the state vector $(y \quad x \quad \varphi \quad v_t)^T$. The observations are the (x, y) coordinates, the azimuth and the tangential velocity:

$$\left.\begin{aligned} z_y(k+1) &= y(k+1) \\ z_x(k+1) &= x(k+1) \\ z_\varphi(k+1) &= \varphi(k+1) \\ z_{v_t}(k+1) &= v_t(k+1) \end{aligned}\right\} \quad (55)$$

in which the azimuth and the tangential velocity usually have the higher data rate than (x, y) have.

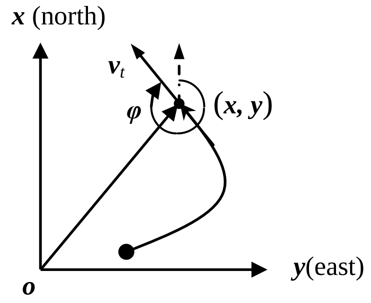

Fig. 1 The 2D mapping frame

Three coefficient matrices in (26) and (27) are

$$A(k) = \begin{pmatrix} 1 & 0 & -v_t(k)\Delta t_{k+1,k}\sin\varphi(k) & \Delta t_{k+1,k}\cos\varphi(k) \\ 0 & 1 & v_t(k)\Delta t_{k+1,k}\cos\varphi(k) & \Delta t_{k+1,k}\sin\varphi(k) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (56)$$

$$B(k) = \begin{pmatrix} \frac{1}{2}\Delta t_{k+1,k}^2\cos\varphi(k) & -\frac{1}{2}\Delta t_{k+1,k}^2\sin\varphi(k) \\ \frac{1}{2}\Delta t_{k+1,k}^2\sin\varphi(k) & \frac{1}{2}\Delta t_{k+1,k}^2\cos\varphi(k) \\ 0 & \Delta t_{k+1,k}^2 / v_t(k) \\ \Delta t_{k+1,k}^2 & 0 \end{pmatrix} \quad (57)$$

$$C(k+1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (58a)$$

Or

$$C(k+1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{(only } \varphi \text{ and } v_t \text{ available)} \quad (58b)$$

where $\Delta t_{k+1,k} = t_{k+1} - t_k$. There are two components in process noise vector associated with $B(k)$: the tangential acceleration $a_t(k)$ and the centrifugal acceleration $a_r(k)$.

Based on (50), a trajectory about 361 meters long was simulated for 40 seconds with the data rate 1 Hz for (x, y) observations and 10 Hz for the azimuth and velocity observations (Fig. 2). The standard deviations used for the observations in the simulation are listed in Table 3. The simulated tangential velocity and azimuth profiles are shown in Fig. 3 and Fig. 4, respectively.

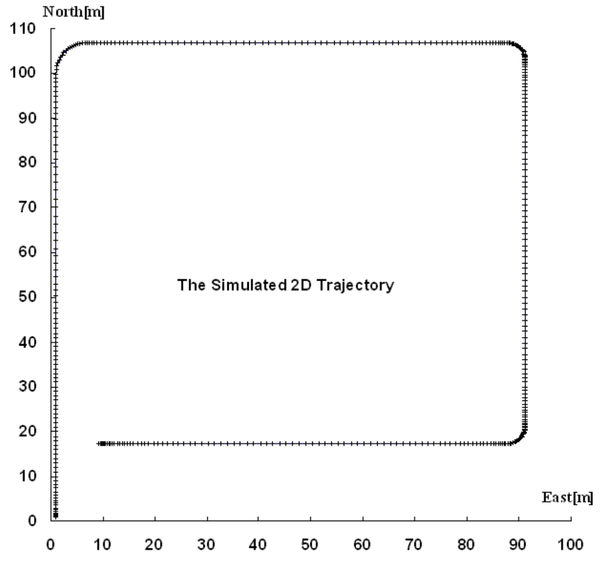

Fig. 2 The simulated 2D vehicle trajectory (40 s)

Table 3 The standard deviations used in the simulation

Standard deviation	Observation	Standard deviation	Process Noise
σ_x , σ_y [m]	0.10	σ_{a_t} [m/s^2]	0.25
σ_φ [deg]	1.00	σ_{a_r} [m/s^2]	0.75
σ_{v_t} [m/s]	0.03		

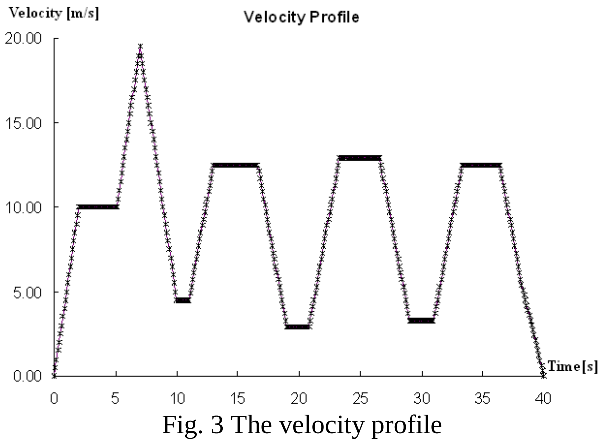

Fig. 3 The velocity profile

The redundancy contribution in groups with respect to the predicted state vector, the process noise vector and the observation vector are shown in Fig. 5. The total redundancy number is equal to 4 if (x, y), φ and v_t are available or 2 if only φ and v_t are available. Fig. 6 gives the redundancy indices of 6 observations and 2 process noise factors. Based on $\delta_0 = 3.42$ as in (14) at the significance level of $\alpha_0 = 1\%$ and with the test power of $1-\beta_0 = 80\%$, their internal reliability is plotted in Fig. 7.

Furthermore, the measure of the external reliability after (28) is given in Fig. 8.

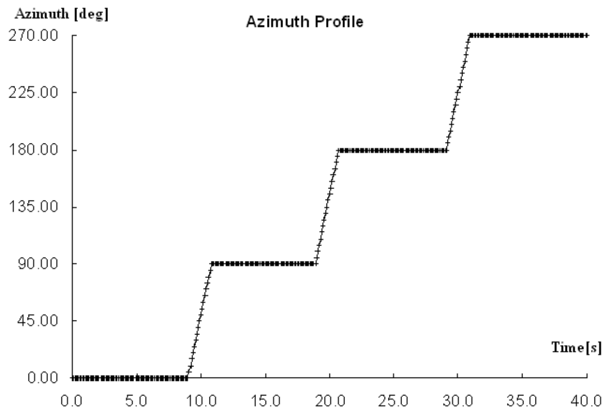

Fig. 4 The azimuth profile

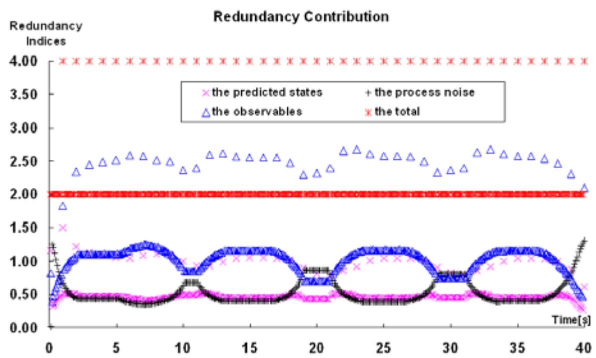

Fig. 5 The redundancy contribution in groups

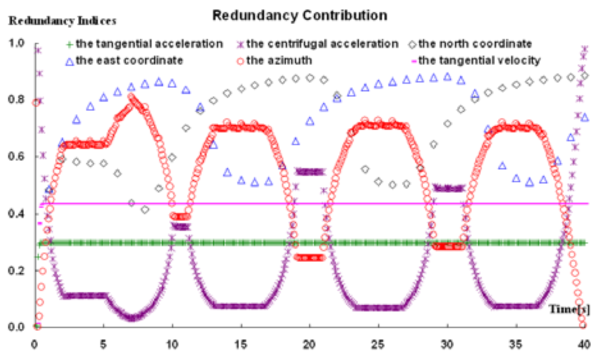

Fig. 6 The redundancy contribution of individual observations and process noise factors

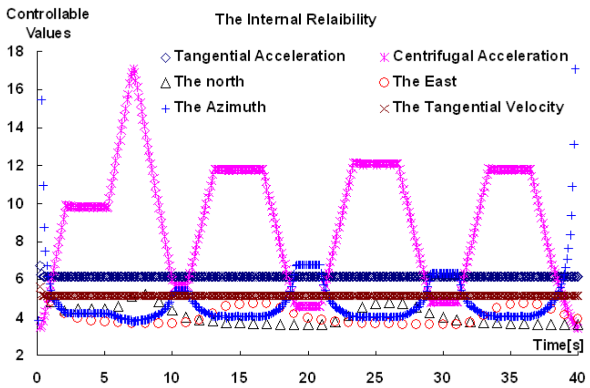

Fig. 7 The internal reliability as in (16)

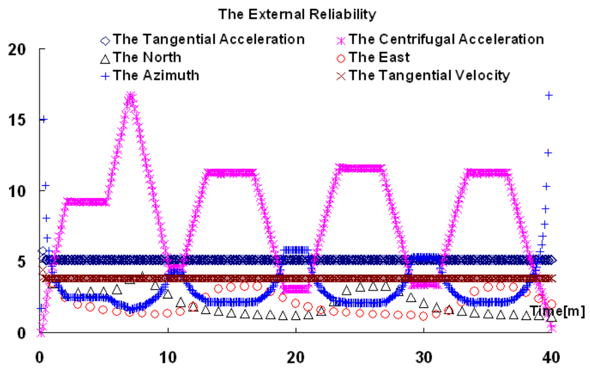

Fig. 8 The external reliability as in (25)

5. Some aspects regarding the use of redundancy contribution in Kalman filtering

This section summarizes three applications about the use of the redundancy contribution in Kalman filtering:

- The simplified algorithm for the variance component estimation in Kalman filtering [Wang, 1997; Caspary & Wang, 1998; Wang, et al, 2009];
- The degrees of freedom of test statistics in Kalman filtering [Wang, 1997, 2008];
- The Robust Kalman filter with the help of the a posteriori variance estimation [Wang, 1997]

5.1. The simplified VCE algorithm in Kalman filtering

The simplified VCE (variance component estimation) algorithm for Kalman filter here means one of the approximate VCE algorithms by [Förstner, 1979], which is just based on the measurement residuals and the measurement redundancy indices through ignoring the non-diagonal elements in the corresponding normal equation system of the rigorous Helmert VCE method.

For an arbitrary epoch $\boldsymbol{k}$, the individual variance factors for the measurements in $\boldsymbol{z}(\boldsymbol{k})$ can be estimated by

$$\hat{\sigma}^2_{z_i z_i}(\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{z_i z_i}(\boldsymbol{k})}{\boldsymbol{r}_{z_i}(\boldsymbol{k})} \qquad (\boldsymbol{i} = 1,\ldots,\boldsymbol{p}) \qquad (59)$$

The accumulative individual variance factor in $\boldsymbol{z}(\boldsymbol{k})$ can be estimated from the past $\boldsymbol{k}$ epochs:

$$\hat{\sigma}^2_{z_i z_i}(1,\ldots,\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{z_i z_i}(1) + \cdots + \boldsymbol{v}^2_{z_i z_i}(\boldsymbol{k})}{\boldsymbol{r}_{z_i}(1) + \cdots + \boldsymbol{r}_{z_i}(\boldsymbol{k})} \quad (\boldsymbol{i} = 1,\ldots,\boldsymbol{p}) \qquad (60)$$

They are corresponding to the diagonal elements of the measurement variance matrix $\boldsymbol{R}(*)$ in (29).

For $\boldsymbol{w}(\boldsymbol{k}-1)$, the similar formulas can be given by:

$$\hat{\sigma}^2_{w_i w_i}(\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{w_i w_i}(\boldsymbol{k})}{\boldsymbol{r}_{w_i}(\boldsymbol{k})} \qquad (\boldsymbol{i} = 1,\ldots,\boldsymbol{m}) \qquad (61)$$

and

$$\hat{\sigma}^2_{w_i w_i}(1,\ldots,\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{w_i w_i}(1) + \cdots + \boldsymbol{v}^2_{w_i w_i}(\boldsymbol{k})}{\boldsymbol{r}_{w_i}(1) + \cdots + \boldsymbol{r}_{w_i}(\boldsymbol{k})} \quad (\boldsymbol{i} = 1,\ldots,\boldsymbol{m}) \qquad (62)$$

As above, they are accordingly the estimates of the diagonal elements of the process noise variance matrix $\boldsymbol{Q}(*)$ in (28).

More specifically about the applications of this VCE algorithm in kinematic positioning can be found in [Wang, 1997; Capsary & Wang, 1998; Wang, et al, 2009].

5.2. Degrees of freedom for test statistics

The test statistics in Kalman filtering can be constructed using either the system innovation sequences or the measurement residuals [Wang, 1997, 2008]. The degrees of freedom of some of the test statistics for measurement residuals may be replaced by the corresponding redundancy indices.

For example, the degrees of freedom in the test statistics for the $\boldsymbol{i}$-th component given in 6.1, 6.2 in [Wang, 2008] are equal to the number of the used measurement residuals. Obviously, if one deals with all of the three measurement vectors given in (35), the total of the degrees of freedom from all of the components at an arbitrary epoch will be much bigger than the number of the total redundant measurements in the system (refer to Section 3.1) at that epoch. In this case, the author suggests using the redundancy indices of the individual measurements to calculate the accumulated degrees of freedom. There is a persuasive argument for doing this. Based on the posteriori variance estimates in (60) or (62), if the number of the used residuals is taken as the degrees of freedom, (60) and (62) become, respectively

$$\hat{\sigma}^2_{z_i z_i}(1,...,\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{z_i z_i}(1) + \cdots + \boldsymbol{v}^2_{z_i z_i}(\boldsymbol{k})}{\boldsymbol{k}} \qquad (63)$$

$$\hat{\sigma}^2_{w_i w_i}(1,...,\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{w_i w_i}(1) + \cdots + \boldsymbol{v}^2_{w_i w_i}(\boldsymbol{k})}{\boldsymbol{k}} \qquad (64)$$

which are definitely smaller than the values from (60) and (62) because the degrees of freedom are here bigger than the ones there. It definitely implies that the statistical critical values are more conservative (bigger) while the corresponding test statistics are too optimistic (smaller).

5.3. The robust Kalman filter with the help of the a posteriori variances

An approach for robust estimation in least squares method with the help of the a-posteriori variance estimation developed by [Li, 1983] was realized in Kalman filtering [Wang, 1997]. Instead of adjusting the weighting functions on the individual measurements, this approach takes the a posteriori estimates of the measurement variances to process the measurements repeatedly.

Let $\boldsymbol{s}$ be the number of the measurement groups, all of the measurements in the same group have the same level of accuracy and uncorrelated each other either in the group or among groups. In order to construct this robust algorithm, one needs three estimated variances: the global a-posteriori variance of weight unit $\hat{\sigma}^2_{g0}(\boldsymbol{k})$ as given by the equation (38) in [Wang, 2008], the global a-posteriori variance of the individual groups of the measurements $\hat{\sigma}^2_{gi}(\boldsymbol{k})$ $(\boldsymbol{i} = 1,...,\boldsymbol{s})$ given by the equation (58) in [Wang, et al, 2009], and the a-posteriori variance of each measurement:

$$\hat{\sigma}^2_{ij}(\boldsymbol{k}) = \frac{\boldsymbol{v}^2_{ij}(\boldsymbol{k})}{\boldsymbol{r}_{ij}(\boldsymbol{k})} \quad (\boldsymbol{i} = 1,...,\boldsymbol{s}; \boldsymbol{j} = 1,...,\boldsymbol{n}_i) \qquad (65)$$

for the $\boldsymbol{j}$-th measurement in the $\boldsymbol{i}$-th group at epoch $\boldsymbol{k}$, wherein $\boldsymbol{r}_{ij}(\boldsymbol{k})$ is the measurement redundant index and $\boldsymbol{n}_i$ is the number of the measurement in the $\boldsymbol{i}$-th group.

The iteration process of this robust algorithm at epoch $\boldsymbol{k}$ runs as follows:

a). Estimate $\hat{\sigma}^2_{g0}(\boldsymbol{k}-1)$ and $\hat{\sigma}^2_{gi}(\boldsymbol{k}-1)$ $(\boldsymbol{i} = 1,...,\boldsymbol{s})$ based on all of the past filtering results from start to the last epoch $\boldsymbol{k}$-1,

b). Filter the data at the current epoch $\boldsymbol{k}$,

c). Compute the initial estimate $[\hat{\sigma}^2_{ij}(\boldsymbol{k})]^{(\tau)}$ ($\tau = 0$ for the initial step) and introduce the $\boldsymbol{F}$-test as

$$\boldsymbol{F}_{ij} = \frac{[\hat{\sigma}^2_{ij}(\boldsymbol{k})]^{(0)}}{\hat{\sigma}^2_{gi}(\boldsymbol{k}-1)} \sim \boldsymbol{F}(1, \boldsymbol{r}_i) \qquad (65)$$

under the null hypothesis: $\boldsymbol{H}_0 : \hat{\sigma}^2_{ij}(\boldsymbol{k}) = \hat{\sigma}^2_{gi}(\boldsymbol{k}-1)$ for $(\boldsymbol{i} = 1,...,\boldsymbol{s})$, where $\boldsymbol{r}_i$ is the accumulated redundancy contribution of the $\boldsymbol{i}$-th measurement group from the past $\boldsymbol{k}$-1 epochs,

d). Adjust the variance:

$$[\hat{\sigma}^2_{ij}(\boldsymbol{k})]^{(\tau+1)} = \begin{cases} \hat{\sigma}^2_{gi}(\boldsymbol{k}-1) & \text{if } \boldsymbol{F}_{ij} \le \boldsymbol{F}_{\alpha}(1, \boldsymbol{r}_i) \\ [\dfrac{\boldsymbol{v}^2_{ij}(\boldsymbol{k})}{\boldsymbol{r}_{ij}(\boldsymbol{k})}]^{(\tau)} & \text{otherwise} \end{cases} \qquad (66)$$

e). Repeat the filter from b) until

$$\left| \hat{x}^{(\tau+1)}_{i_x}(\boldsymbol{k}) - \hat{x}^{(\tau)}_{i_x}(\boldsymbol{k}) \right| < \varepsilon \qquad (\boldsymbol{i}_x = 1,...,\boldsymbol{n}) \qquad (67)$$

for all of the components in the state vector,

f). Go on the next epoch starting from a).

More specific details about this robust algorithm refer to [Li, 1983; Wang, 1997]. The numerical examples for this robust Kalman filter can be found in [Wang, 1997].

6. Concluding Remarks

Based on the standard model given in 3.1, the mathematical and statistical fundamental of the reliability analysis was systematically described for the Kalman filter algorithm in this manuscript. The alternate formulation of Kalman filter made it possible to distinctly derive the redundancy contribution for the Kalman filter algorithm. This delivery allows users to perform the reliability analysis in Kalman filtering exactly in the same way as it has been done in the least squares estimation. A numerical example was provided for the reliability analysis on kinematic data processing in Kalman filtering in Section 4. Additionally, a summary of three interesting applications: the simplified VCE algorithm, degrees of freedom for test statistics and the robust Kalman filter with the help of the a posteriori variances were given regarding the potential use of the redundancy contribution in Kalman filtering.

Furthermore, as the author pointed out in [Wang, 2008], the alternate formulation of Kalman filter algorithm makes it possible to statistically conduct the system diagnosis epochwise against different error sources because three types of the available stochastic information: the current available measurements, the current process noise and the one step predicted states, are kept separately. So far either the system innovation sequences or the measurements and the predicted states blended with the process noise have commonly been used for quality control in Kalman filtering.

7. References

Baarda, Willem (1968). **A Testing Procedure for Use in Geodetic Networks**. No. 5, Vol. 2, Publication on Geodesy, Netherlands Geodetic Commission, Delft, Netherlands, 1968.

Barber, Katherine (1998). **The Canadian Oxford Dictionary**, Oxford University Press, Canada 1998.

Caspary, Wilhelm (1988). **Concepts of Network and Deformation Analysis**. Monograph 11, School of Surveying, University of New South Wales, Kensington, N.S.W., Australia, ISBN 0-85839-044-2, August 1988.

Caspary, Wilhelm; Wang, Jianguo (1998). **Redundanzanteile und Varianzkomponenten im Kalman Filter**. Vol.123, No.4, 1998, pp.121-128.

Chen, Yongqi; Wang, Jinling (1996). **Reliability Measures for Correlated Observations**. Zeitschrift für Vermessungwesen, Vol. 121, No. 5, 1996, pp. 211-219.

Chui, C.K.; Chen, G. (2009): ***Kalman Filtering with Real-Time Applications,*** Fourth edition, ISBN 978-3-540-87848-3, Springer-Verlag Berlin Heidelberg, 2009.

Eichhorn, Andreas (2005). **Ein Beitrag zur Identifikation von dynamischen Strukturmodellen mit Methoden der adaptiven Kalman-filterung**. Deutsche Geodätische Kommission (DGK), Series C, No. 585, Munich 2005.

Förstner, W. (1979): **Ein Verfahren zur Schätzung von Varianz- und Kovarianzkomponenten**. AVN 86 (11-12), 1979, pp. 446-453.

Gao, Yang (1992). **A Robust Quality Control System for GPS Navigation and Kinematic Positioning**. PhD. Thesis, University of Calgary, ISBN: 0315831480, Calgary, September 1992.

Gelb, Arthur et al. (1974). **Applied Optimal Estimation**. The Analytic Sciences Corporation & The M.I.T. Press, Cambridge Massachusetts London, 1974.

Hahn, M.; van Mierlo, J. (1986). **Statistische Grundlagen für die Analyse von Ausgleichungsergebnissen**, DVW-Mitteilungen, Baden-Württemberg, Special Issue – Evaluation of geodetic Networks, pp. 22-46.

Hewitson, Steve (2006). **Quality Control for Integrated GNSS and Inertial Navigation Systems**, PhD. Thesis, University of New South Wales, Sydney, July 2006.

Jäger, R.; Bill, R. (1986). **Genauigkeits- und Zuverlässigkeits-maße in Verdichtungsnetzen**. DVW-Mitteilungen, Baden-Württemberg, Special Issue – Evaluation of geodetic Networks, pp. 70-98

Jia, M. and Ding, X., and Montgomery, B., (1998). **On reliability measures for kinematic surveys**, *Geomatica*, Vol. 52, No.1, pp.37-44.

Leick, Alfred (2004). **GPS Satellite Surveying**. John Wiley & Sons, Inc., ISBN 0-471-05930-7, Hoboken, New Jersey, 2004.

Li, Deren (1993). **Ein Verfahren zur Aufdeckung grober Fehler mit Hilfe der a-posteriori-Varianz Schätzung.** Bildmessung und Luftbildwesen, Vol. 51, No. 5, 1983, pp. 184-187.

Li, Deren; Yuan, Xiuxiao (2002). **Error Processing and Reliability Theory**. Wuhan University Press, ISBN 7-307-03474-3, Wuhan, 2002 (in Chinese).

Lu, Gang (1991). **Quality control for Differential Kinematic GPS Positioning**. MSc. Thesis, UCGE Reports No. 20042, University of Calgary, Calgary, August 1991.

Mehra, R. K.; Peschon, J. (1971). **An Innovations Approach to Fault Detection and Diagnosis in Dynamic Systems**. Automatica, Vol. 16, 1971, S. 637–640.

Ramm, Katrin (2008). **Evaluation von Filter-Ansätzen für die Positionsschätzung von Fahrzeugen mit den Werkzeugen der Sensitivitätsanalyse**. Deutsche Geodätische Kommission (DGK), Series C, No. 619, Munich 2008.

Salzmann, Martin. (1991). **MDB**: **A Design tool for integrated navigation systems**. Bulletin Géodésique (1991) 65, pp. 109-115.

Salzmann, Martin; Teunissen, P.J.G. (1989). **Quality Control in Kinematic Data Processing**. Land Vehicle Navigation 1989, DGON, Verlag TÜV Rheinland, Köln, 1989, pp. 355–366.

Salzmann, Martin (1993). **Least Squares Filtering and Testing for Geodetic Navigation Applications**. Number 37, Publication on Geodesy, Netherlands Geodetic Commission, Delft, 1993.

Stöhr, M. (1986). **Der Kalman–Filter und seine Fehlerprozeße unter besonderer Berücksichtigung der Auswirkung von Modellfehlern**. Forschung–Ausbildung–Weiterbildung Bericht Heft 19, Universität Kaiserlautern, Fachbereich Mathematik, August 1986.

Tiberius, C.C.J.M. (1998). **Recursive data processing for kinematic GPS surveying,** PhD thesis**,** Delft University of Technology, ISBN 90-80414701-9, Delft, July 1998.

Wang, Jian-Guo; Gopaul, Nilesh; Scherzinger, Bruno (2009). **Practical Algorithms of Variance Component Estimation for Static and Kinematic GPS single Point Positioning**. International Workshop on Geodetic Theory, Shanghai, June 1-2, 2009.

Wang, Jian-Guo (2008). **Test Statistics in Kalman Filtering**. Vol. 7, No. 1, Journal of GPS, pp. 81-90.

Wang, Jianguo (1997). **Filtermethoden zur fehlertoleranten kinematischen Positions-bestimmung**. Schritenreihe Studiengang Vermessungswesen, Federal Arm-Forced University Munich, Germany, No. 52, Neubiberg, 1997.

Willsky, Alan S., Deyst, J. J.; Crawford, B. S. (1974). **Adaptive Filtering and Self–Test Methods for Failure Detection and compensation**. Proceedings of the 1974 JACC, Austin, Texas, June 19–21, 1974.

Willsky, Alan S. (1976). **A Survey of Design Methods for Failure Detection in Dynamic Systems**. Automatica, Vol. 12, 1976, pp. 601–611.