Journal of Global Positioning Systems (2002)
Vol. 1, No. 2: 122-131

Journal of Global Positioning Systems

A General Criterion of Integer Ambiguity Search

Guochang Xu

GeoForschungsZentrum Potsdam (GFZ), Telegrafenberg, 14473 Potsdam, Germany

Received: 9 October 2002 / Accepted: 6 January 2003

Abstract. A general criterion for integer ambiguity searching is derived in this paper. The criterion takes into account not only the residuals caused by ambiguity parameter changing, but also the residuals caused by coordinates changing through ambiguity fixing. The search can be carried out in a coordinate domain, in an ambiguity domain or in both domains. The three searching scenarios are theoretically equivalent. The optimality and uniqueness properties of the proposed criterion are also discussed. A numerical explanation of the general criterion is outlined. The theoretical relationship between the general criterion and the commonly used least squares ambiguity search (LSAS) criterion is derived in an equivalent case in detail. It shows that the LSAS criterion is just one of the terms of the equivalent criterion. Numerical examples are given to illustrate the behaviour of the two components of the equivalent criterion.

Key words: Integer Ambiguity Searching Criterion

---

1 Introduction

It is well-known that the ambiguity resolution is a key problem which has to be solved in GPS static and kinematic precise positioning. Some well-derived ambiguity fixing and searching algorithms have been published during the last decade. These methods can be generally classified as four types. The first type includes Remondi's static initialisation approach (Remondi 1984; Hofmann-Wellenhof et al. 1997; Wang et al. 1988), which requires a static survey time to solve the ambiguity unknowns after any complete loss of lock. Normally, the results are good enough to take a round up ambiguity fixing. The second type includes the so-called phase-code combined methods (Han & Rizos 1995, 1997; Sjoeberg 1998, 1999); the phase and code have to be used in the derivation as if they have the same precision, and in case of anti-spoofing (AS), the C/A code has to be used. A search process is still needed in this case. The third type is the so-called ambiguity function method (Remondi 1984; Hofmann-Wellenhof et al. 1997); its search domain is a geometric one. The fourth type includes approaches, their search domain is only in domain of ambiguity, including some optimal algorithms to reduce the search area and to accelerate the search process (Euler & Landau 1992; Teunissen 1995; Leick 1995; Han & Rizos 1995, 1997). Because of the statistic character of validation criteria, sometimes no valid result is obtained at the end of the search processes.

The effort to develop the KSGSoft (Kinematic/Static GPS Software) at the GeoForschungsZentrum (GFZ) Potsdam began at the beginning of 1994 due to the requirement of kinematic GPS positioning in aerogravimetry applications (Xu et al. 1998, 1999). An optimal ambiguity resolution method is needed in order to implement it into the software; however, selecting the published algorithms has turned out to be a difficult task. This has led to the independent development of this so-called integer ambiguity search method (cf. Xu 2003). It turns out to be a very promising algorithm; the search domain could be in the domain of coordinate or ambiguity or both, and it is reliable and fast. Using this general criterion, an optimal ambiguity vector can be searched for and found out. The searched result is the optimal one under the least squares principle and integer ambiguity property.

The theoretical background of this method is the well-known conditional least squares adjustment and will be outlined below in the section 2. The well-known least squares ambiguity search (LSAS) criterion is derived in section 3. An analogue derivation of using coordinate condition is outlined in section 4. A general criterion is presented in section 5. Properties of the general criterion are discussed in section 6. The relationship between the general criterion and the least squares ambiguity search criterion is derived in an equivalent case in section 7.

Numerical examples are given in section 8. Conclusions and comments are given in the last section.

2 Conditional Least Squares Adjustment

The principle of least squares adjustment with condition equations can be summarised as below (Cui et al. 1982; Leick 1995; Gotthardt 1978; Xu 2003):

1). Linearised observation equation system can be represented by:

$$V = L - AX, \qquad P \tag{1}$$

where

- L: observation vector of dimension m,
- A: coefficient matrix of dimension m × n,
- X: unknown vector of dimension n,
- V: residual vector of dimension m,
- n: number of unknowns,
- m: number of observations,
- P: symmetric and quadratic weight matrix of dimension m × m.

2). The condition equation system can be written as:

$$CX - W = 0 \tag{2}$$

where

- C: coefficient matrix of dimension r × n,
- W: constant vector of dimension r,
- r: number of conditions.

3). The least squares criterion for solving the observation equations with condition equations is well-known as:

$$V^TPV = min \tag{3}$$

where

- V^T: the transpose of the related vector V.

4). The solution of the conditional problem (1) and (2) under the least squares principle (3) is then:

$$X_c = (A^TPA)^{-1}(A^TPL) - (A^TPA)^{-1}C^TK$$
$$= (A^TPA)^{-1}(A^TPL - C^TK) \tag{4}$$

and

$$K = (CQC^T)^{-1}(CQW_1 - W) \tag{5}$$

where A^T, C^T are the transpose matrices of A, C, the superscript $^{-1}$ is an inversion operator, $Q = (A^TPA)^{-1}$, K is a gain vector (of dimension r), index $_c$ is used to denote the variables related to the conditional solution, and $W_1 = A^TPL$.

5). The accuracies of the solutions are then:

$$p[i] = s_d sqrt(Q_c[i][i]) \tag{6}$$

where i is the element index of a vector or a matrix, sqrt() is the square root operator, s_d is the standard deviation (or sigma) of unit weight, $p[i]$ is the i-th element of the precision vector, $Q_c[i][i]$ is the i-th diagonal element of the quadratic matrix Q_c, and

$$Q_c = Q - QC^TQ_2CQ \tag{7}$$

$$Q_2 = (CQC^T)^{-1} \tag{8}$$

$$s_d = sqrt((V^TPV)_c/(m-n+r)) \qquad if(m > n-r) \tag{9}$$

6). For recursive convenience, $(V^TPV)_c$ can be calculated by using:

$$(V^TPV)_c = L^TPL - (A^TPL)^TX_c - W^TK \tag{10}$$

Above are the complete formulas of conditional least squares adjustment. The application of such an algorithm for the purpose of integer ambiguity search will be further discussed in later sections.

3 Integer Ambiguity Search in Ambiguity Domain

GPS observation equations can be represented with (1). Considering the case without conditions (2), i.e., $C = 0$ and $W = 0$, the above equations are the same as the results of normal least squares adjustment. So the least squares solution of (1) is

$$X_0 = Q(A^TPL) = QW_1 \tag{11}$$

and

$$(V^TPV)_0 = L^TPL - (A^TPL)^TX_0 \tag{12}$$

$$s_d = sqrt((V^TPV)_0/(m-n)), \qquad if(m > n) \tag{13}$$

$$p[i] = s_d sqrt(Q[i][i]) \tag{14}$$

Where index $_0$ is used for convenience to denote the variables related to the normal least squares solution without conditions. X_0 is the complete unknown vector including coordinates and ambiguities and is called a float solution later on. Solution X_0 is the optimal one under least squares principle. However, because of the observation and model errors as well as method limitations, float solution X_0 may not be exactly the right one, e.g the ambiguity parameters are real numbers and do not fit to the integer property. Therefore one sometimes needs to search for a solution, say X, which not only fulfils some special conditions, but also meanwhile keeps the deviation of the solution as small as possible (minimum). This can be represented by

$$V_x^TPV_x = min \tag{15}$$

In (15) the V_x is the residuals vector in case of solution X. For simplification, let:

$$X = \begin{pmatrix} Y \\ N \end{pmatrix} \qquad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

$$W_1 = A^T PL = \begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix}$$

$$M = A^T PA = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \qquad M = Q^{-1} \tag{16}$$

where Y is the coordinate vector, N is the ambiguity vector (generally, a real vector). To use the conditional adjustment algorithm for integer ambiguity searching in ambiguity domain, the condition shall be selected as N = W, here W is, of course, an integer vector. Generally, letting $C = (0, E)$, then condition (2) turns out to be:

$$N = W \tag{17}$$

Using definitions of C and Q, one has:

$$CQ = \begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix}$$

$$CQC^T = Q_{22}$$

The float solution is denoted as

$$X_0 = \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} Q_{11}W_{11} + Q_{12}W_{12} \\ Q_{21}W_{11} + Q_{22}W_{12} \end{pmatrix}$$

where X_0 is the solution of (1) without condition (17). The gain vector K_N can be computed by:

$$K_N = (Q_{22})^{-1}(CQW_1 - W) = (Q_{22})^{-1}(N_0 - W) \tag{18}$$

So under the condition (17), the conditional least squares solution (4) can be written as:

$$X_c = \begin{pmatrix} Y_c \\ N_c \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} W_{11} \\ W_{12} - K_N \end{pmatrix}$$

$$= \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} - \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} K_N \tag{19}$$

Simplifying (19), one gets:

$$Y_c = Y_0 - Q_{12}K_N \tag{20}$$

and

$$N_c = N_0 - Q_{22}K_N = N_0 - Q_{22}(Q_{22})^{-1}(N_0 - W) = W \tag{21}$$

The precision computing formulas under condition (17) can be derived as below:

$$Q_c = Q - QC^T(Q_{22})^{-1}CQ$$

$$= \begin{pmatrix} Q_{11} - Q_{12}(Q_{22})^{-1}Q_{21} & 0 \\ 0 & 0 \end{pmatrix} \tag{22}$$

$$(V^T PV)_c = (V^T PV)_0$$

$$+ (N_0 - W)^T (Q_{22})^{-1}(N_0 - W) \tag{23}$$

where $(V^TPV)_0$ is the value obtained without condition (17). The second term on the right-hand side of (23) is the often-used least squares ambiguity search criterion, (cf. e.g. Teunissen 1995; Euler & Landau 1992; Hofmann-Wellenhof et al. 1997), which can be expressed as

$$\delta(dN) = (N_0 - N)^T(Q_{22})^{-1}(N_0 - N) \tag{24}$$

It indicates that any ambiguity fixing will cause an enlargement of the standard deviation. However, one may also notice that here only the enlargement of the standard deviation caused by ambiguity parameter changing has been considered. Any ambiguity fixing will lead to a related coordinate changing (cf. (20)). Furthermore, the condition (17) does not really exist. Ambiguities are integers, however, they are unknowns. The formula to compute the accuracy vector of the ambiguity does not exist too, because the ambiguity condition is considered exactly known in conditional adjustment (cf. Xu 2003).

4 Standard Deviation Enlargement Caused by Coordinate Changing

Analogous to above discussion, the condition could be selected as Y = W, here W is a coordinate vector. Generally, letting C = (E, 0), where E is an identity matrix with dimension of $r \times r$, C has dimension $r \times n$, condition (2) turns out to be:

$$Y = W \tag{25}$$

Using definitions of C and Q, one has:

$$CQ = \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix}$$

$$CQC^T = Q_{11}$$

The float solution is denoted by

$$X_0 = \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} Q_{11}W_{11} + Q_{12}W_{12} \\ Q_{21}W_{11} + Q_{22}W_{12} \end{pmatrix}$$

where X_0 is the solution of (1) without condition (25). The gain vector K_Y can be computed by using (5):

$$K_Y = (Q_{11})^{-1}(CQW_1 - W) = (Q_{11})^{-1}(Y_0 - W) \tag{26}$$

So under the condition (25), the conditional least squares solution (4) can be written as:

$$X_c = \begin{pmatrix} Y_c \\ N_c \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} W_{11} - K_Y \\ W_{12} \end{pmatrix}$$

$$= \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} - \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} K_Y \tag{27}$$

Simplifying (27), one gets:

$$Y_c = Y_0 - Q_{11}K_Y$$

$$= Y_0 - Q_{11}(Q_{11})^{-1}(Y_0 - W) = W \tag{28}$$

and

$$N_c = N_0 - Q_{21}K_Y \quad (29)$$

For any given constant coordinate vector W, an ambiguity vector N_c can be found out (or computed). In such a case, N_c is a float vector and Y_c is exactly the same as that given in condition (25). If the Y_c is a correct one, the computed N_c should be very close to an integer vector under the assumptions made at the beginning. The searched integer ambiguity vector is then Fix(N_c), where Fix() is a round up function for rounding up a real number to its nearest integer number. A more detailed discussion on the use of the rounding function to the computed vector N_c will be made in section 6. The precision computing formula under condition (25) can be derived by using definitions:

$$Q_c = Q - QC^T(Q_{11})^{-1}CQ$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} - Q_{21}(Q_{11})^{-1}Q_{12} \end{pmatrix} \quad (30)$$

$$(V^TPV)_c = (V^TPV)_0$$

$$+ (Y_0 - W)^T(Q_{11})^{-1}(Y_0 - W) \quad (31)$$

where $(V^TPV)_0$ is the value obtained without condition (25). The second term on the right-hand side of (31) cannot be used directly as a criterion for an ambiguity search; however, it indicates that any coordinate change will cause an enlargement of the standard deviation. For convenience, we denote

$$\delta_I(dY) = (Y_0 - Y)^T(Q_{11})^{-1}(Y_0 - Y) \quad (32)$$

It is obvious that such an effect has to be taken into account in the ambiguity fixing. This will be further discussed in next section.

5 Integer Ambiguity Search in Coordinate and Ambiguity Domains

Even the to be fixed solution is an unknown vector, however, in order to see the enlargement of the standard deviation caused by the fixed solution, the condition could be selected as $X = W$, here W consists of two sub-vectors (coordinate and ambiguity parameter related sub-vectors). And only the ambiguity parameter related sub-vector is an integer one. Letting $C = E$, condition (2) is then:

$$X = W \quad (33)$$

One has:

$$CQ = CQC^T = Q$$

Denote $X_0 = QW_1$; here X_0 is the solution of (1) without condition (33). The gain K can be computed by:

$$K = Q^{-1}(CQW_1 - W) = Q^{-1}(X_0 - W) \quad (34)$$

So under the condition (33), the conditional least squares solution (4) can be written as:

$$X_c = X_0 - QK = X_0 - QQ^{-1}(X_0 - W) = W \quad (35)$$

Precision computing formulas under condition (33) can be derived as below:

$$Q_c = 0 \quad (36)$$

$$(V^TPV)_c = (V^TPV)_0 + (X_0 - W)^TQ^{-1}(X_0 - W) \quad (37)$$

where $(V^TPV)_0$ is the value obtained without condition (33).

The second term on the right side of (37) can be used as a general criterion for integer ambiguity search, i.e.:

$$\delta = (X_0 - X)^TQ^{-1}(X_0 - X) \quad (38)$$

It indicates the enlargement of the standard deviation caused by fixed solution X. A minimum value of (38) is equivalent to a minimum value of $(V^TPV)_c$. Therefore an optimal fixed solution has to be searched for so that (38) has the minimum value. To be noticed is that the minimum value of (38) is not a minimization process, but just a searching process to find out the optimal X. (38) has obviously a more general form than the least squares ambiguity search criterion (24) does.

In all above three derivations, to be noticed is that in the precision vector the condition related elements are not defined. This is because in the conditional adjustment conditions are considered exactly known. However, in integer ambiguity searching, to be tested candidates (e.g. integer ambiguity) are indeed not exactly known or say, known with uncertainty (float solution with its precision). The uncertainty of the computed ambiguity and selected coordinate vectors (related to the searching in coordinate domain), and the uncertainty of the computed coordinate and selected ambiguity vectors (related to the searching in ambiguity domain), as well as the uncertainty of the selected vector in both domains (related to the searching in both domains) should be taken into account in any cases. Therefore (38) is a more reasonable criterion and should be used generally in ambiguity searching no matter in which domain the search will be made. Under such criterion, the deviation of the result vector X related to the float vector X_0 is homogenously considered. For computing the precision of the searched X, the formulas of least squares adjustment shall be further used, and meanwhile the enlarged residuals shall be taken into account by

$$p[i] = s_d\, sqrt(Q[i][i]) \quad (39)$$

$$s_d = sqrt((V^TPV)_c/(m-n)) \qquad if(m > n) \quad (40)$$

$$(V^TPV)_c = (V^TPV)_0 + \delta \quad (41)$$

In other words, the original Q matrix and $(V^TPV)_0$ of the least squares problem (1) are further used. The δ has the function of enlarging the standard deviation. The formulas of (38), (39--41) are partly derived from the conditional adjustment, however, the formulas have nothing to do with the conditions. Searching for a minimum δ leads to a minimum of s_d and therefore the best precision vector $p[i]$. The geometric explanation of here proposed integer ambiguity searching criterion is discussed in section 6.

The general criterion of (38) is used for all three searching scenarios, where X_0 is the float solution, Q is the inversion of the complete normal matrix of (1). X is the selected candidate vector in case of searching in both coordinate and ambiguity domains. In case of searching in coordinate domain, X consists of the selected sub-vector of Y_c in (28) and the computed sub-vector of Fix(N_c) in (29), i.e.:

$$X = \begin{pmatrix} Y_c \\ Fix(N_c) \end{pmatrix} \tag{42}$$

The reason why the Fix(N_c) is used here will be discussed theoretically in next section. In the case of searching in ambiguity domain, X consists of the selected sub-vector of N_c in (21) and the computed coordinate sub-vector Y_c in (20), i.e.:

$$X = \begin{pmatrix} Y_c \\ N_c \end{pmatrix} \tag{43}$$

6 Properties of the General Criterion

1). Equivalence of the Three Searching Processes

To be emphasised is that the same searching criterion (38) and the same formulas of precision estimation (39—41) are used in the three integer ambiguity search scenarios. And the same normal equations of (1) is used to compute the vector N_c using selected Y_c or to compute the Y_c using selected N_c if necessary. The three searching processes indeed deal with the same problem, just as different ways of searching are used.

Suppose by searching in ambiguity domain, the vector $X = (Y_c \;\; N_c)^T$ is found so that δ reaches the minimum, where N_c is selected integer sub-vector and Y_c is the computed one. In the case of searching in coordinate domain, if the selected coordinate sub-vector Y is exactly the same as Y_c, then integer sub-vector N obtained by computation should be exactly the same as N_c. Taking the computing errors into account, the computed N could be a real vector, however, the errors must be very small and the rounding vector Fix(N) must be the same as N_c. (This is also the reason why the rounding function is used for the computed vector N_c in the case of search in coordinate domain). We see now the same results will be obtained theoretically in the both searching cases. Therefore, the searching methods in coordinate domain or in ambiguity domain are theoretically equivalent.

Suppose by searching in ambiguity domain, again, the vector $(Y_c \;\; N_c)^T$ is obtained. And in the case of searching in both coordinate and ambiguity domains, a candidate vector $X = (Y \;\; N)^T$ is selected so that δ reaches the minimum, where N is selected integer sub-vector and Y is selected coordinate vector. Because of the optimality and uniqueness properties of the vector X in (38) (please refer to 2, which is discussed next), here selected $(Y \;\; N)^T$ must be equal to $(Y_c \;\; N_c)^T$. So the theoretical equivalency of the three searching processes is confirmed.

In practice, it could be difficult to have a selected Y that exactly equals the computed Y_c (computed by searching in ambiguity domain). However, it is always possible to get a Y that is as close as required to Y_c by selecting smaller search steps.

2). Optimality and Uniqueness Properties

The float solution X_0 is the optimal and unique solution of (1) under the least squares principle. Using the integer ambiguity search criterion (38), analogously, the searched vector X is the optimal solution of (1) under the least squares principle and integer ambiguity properties. A minimum of δ in (38) will lead to a minimum of $(V^TPV)_c$ in (41). The uniqueness property is obvious. If X_1 and X_2 are such that $\delta(X_1) = \delta(X_2) = min.$, or $\delta(X_1) - \delta(X_2) = 0$, then by using (38), one may assume that X_1 must be equal to X_2.

3). Geometric Explanation of the General Criterion

Geometrically, $\delta = (X_0 - X)^T Q^{-1} (X_0 - X)$ is the "distance" between the vector X and float vector X_0. The distance contributed to enlarge the standard deviation s_d (cf. (40)). Ambiguity searching is then the search for the vector, which own the integer ambiguity property and has the minimum distance to the float vector.

In the next section, the relationship between above proposed general criterion and the common used least squares ambiguity search criterion (derived in §3) will be discussed.

7 Relationship Between the Two Criteria

We are going to prove theoretically that LSAS criterion (24) is just one of the terms of an equivalent criterion of the general criterion (38) as follows.

The normal equation of (1) can be denoted by (use notation of (16)):

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} Y \\ N \end{pmatrix} = \begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix} \quad (44)$$

or

$$M_{11}Y + M_{12}N = W_{11} \quad (45)$$

$$M_{21}Y + M_{22}N = W_{12} \quad (46)$$

The normal equation (45) and (46) can be solved by block-wise elimination as follows. From (45), one has:

$$Y = (M_{11})^{-1}(W_{11} - M_{12}N)$$

Setting Y into (46), one gets a normal equation related to the second block of unknowns:

$$M_2N = B_2 \quad (47)$$

where

$$M_2 = M_{22} - M_{21}(M_{11})^{-1}M_{12} \quad (48)$$

$$B_2 = W_{12} - M_{21}(M_{11})^{-1}W_{11} \quad (49)$$

Similarly, from (46), one has

$$N = (M_{22})^{-1}(W_{12} - M_{21}Y) \quad (50)$$

Setting N into (45), one gets a normal equation related to the first block of unknowns:

$$M_1Y = B_1 \quad (51)$$

where

$$M_1 = M_{11} - M_{12}(M_{22})^{-1}M_{21} \quad (52)$$

$$B_1 = W_{11} - M_{12}(M_{22})^{-1}W_{12} \quad (53)$$

Then the normal equation of (44) can be written by combining (51) and (47) as

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} Y \\ N \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad (54)$$

(44) and (54) are two equivalent normal equations, therefore the integer ambiguity search using (44) or (54) are also equivalent. Using the notation (16), the normal equation of (1) is $MX = W_1$ and the general criterion is (38). Because of $M = Q^{-1}$, (38) is the same as: $(X_0 - X)^TM(X_0 - X)$. So for the normal equation of (54), the related general criterion (38) turns out to be (put the diagonal M into above formula!):

$$\delta_I = \begin{pmatrix} Y_0 - Y \\ N_0 - N \end{pmatrix}^T \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} Y_0 - Y \\ N_0 - N \end{pmatrix}$$

or

$$\delta_I = (Y_0 - Y)^TM_1(Y_0 - Y) + (N_0 - N)^TM_2(N_0 - N) \quad (55)$$

It has to be emphasised that search criterion (55) is equivalent to the criterion (38), however, they are not identical, or generally, $\delta \neq \delta_I$. Furthermore, denote

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \quad (56)$$

where (cf. e.g. Cui et al. 1982; Leick 1995; Gotthardt 1978)

$$Q_{11} = (M_{11} - M_{12}(M_{22})^{-1}M_{21})^{-1} \quad (57)$$

$$Q_{22} = (M_{22} - M_{21}(M_{11})^{-1}M_{12})^{-1} \quad (58)$$

$$Q_{12} = (M_{11})^{-1}(-M_{12}Q_{22}) \quad (59)$$

$$Q_{21} = (M_{22})^{-1}(-M_{21}Q_{11}) \quad (60)$$

then after comparing (57) and (58) with (52) and (48) one has

$$Q_{11} = (M_1)^{-1}, \quad Q_{22} = (M_2)^{-1}$$

or

$$M_1 = (Q_{11})^{-1}, \quad M_2 = (Q_{22})^{-1} \quad (61)$$

Then (55) turns out to be

$$\delta_I = (Y_0 - Y)^T(Q_{11})^{-1}(Y_0 - Y) + (N_0 - N)^T(Q_{22})^{-1}(N_0 - N) \quad (62)$$

Note that the second term on the right-hand side of (62) is exactly the same as the criterion of the least squares ambiguity search (24). In other words, the criterion of least squares ambiguity search is just one term of the equivalent criterion (62) (q.e.d).

It should be emphasised that the consistency between the coordinate sub-vector Y and ambiguity sub-vector N is implicitly used by the proof. Therefore (62) is only valid if the Y and N are consistent each other. The first term on the right-hand side of (62) is the same as the (32), which indicates an enlargement of the standard deviation due to the coordinate change caused by ambiguity fixing.

Now, it is obvious that

1). only if one may lead from a minimum value of (24)

$$\delta(dN) = (N_0 - N)^T(Q_{22})^{-1}(N_0 - N) \quad (63)$$

to get a minimum value of (62)

$$\delta_I = (Y_0 - Y)^T(Q_{11})^{-1}(Y_0 - Y) + (N_0 - N)^T(Q_{22})^{-1}(N_0 - N) \quad (64)$$

then the least squares ambiguity search is equivalent to the general method proposed in §5. However, such a generality does not exist. Therefore, the LSAS criterion is generally not equivalent to the criterion (64) (which is equivalent to the general criterion (38)). Furthermore, using (20) and (18) one has

$$Y_0 - Y = Q_{12}(Q_{22})^{-1}(N_0 - N) \quad (65)$$

Putting (65) into (64), one has

$$\delta_1 = (N_0 - N)^T \cdot \{(Q_{22})^{-1}[E + Q_{21}(Q_{11})^{-1}Q_{12}(Q_{22})^{-1}]\} \cdot (N_0 - N) \quad (66)$$

One may see clearly now the differences between the two criteria (63) and (66).

2). If one may not lead from a minimum value of (63) to get a minimum value of (64), then the least squares ambiguity search may not find the optimal results in view point of the criterion (62). In this case, only criterion (62) reaches a minimum with a unique and optimal vector *X*.

3). The coordinate change due to the ambiguity fixing has not been taken into account in the least squares ambiguity search criterion.

A by-product of above derivation is that we have now a criterion (62) which is equivalent to the criterion (38). By computing the precision vector of (39)—(41), the δ has to be computed using (38), because the δ is not equal δ_l in general.

8 Numerical Examples of General Criterion and LSAS Criterion

Several numerical examples are given here to illustrate the behaviour of the two terms of the criterion. For convenience, we denote the first and second terms of the right-hand side of (62) as $\delta(dY)$ and $\delta(dN)$ respectively. $\delta_l = \delta(dY)+ \delta(dN)$ is the equivalent criterion of the general criterion and is denoted as $\delta(total)$. The term $\delta(dN)$ is the LSAS criterion. Of course, the search is made in the ambiguity domain. Analogues, the general criterion is also used for search in the ambiguity domain. The search area is determined by the precision vector of the float solution. All possible candidates are tested one by one, and the related δ_l are compared to each other to find out the minimum.

In the first example, precise orbits and dual-frequency GPS data of 15 April 1999 at station Brst (N 48.3805°, E 355.5034°) and Hers (N 50.8673°, E 0.3363°) are used. Session length is 4 hours. The total search candidate number is 1020. Results of the two sigma components are illustrated as 2-D graphics with the 1st axis of search number and the 2nd axis of sigma in Fig. 1. The red and blue lines represent $\delta(dY)$ and $\delta(dN)$, respectively. $\delta(dY)$ reaches the minimum at the search number 237, and $\delta(dN)$ at 769. $\delta(total)$ is plotted in Fig. 2, and it shows that the general criterion reaches the minimum at the search number 493. For more detail, a part of the results are listed in the Tab. 1.

Tab. 1 Sigma values of searching process

Search No.	δ(dN)	δ(dY)	δ(total)
237	183.0937	97.8046	280.8984
493	181.7359	97.9494	279.6853
769	93.3593	315.2760	408.6353
771	96.0678	343.5736	439.6414

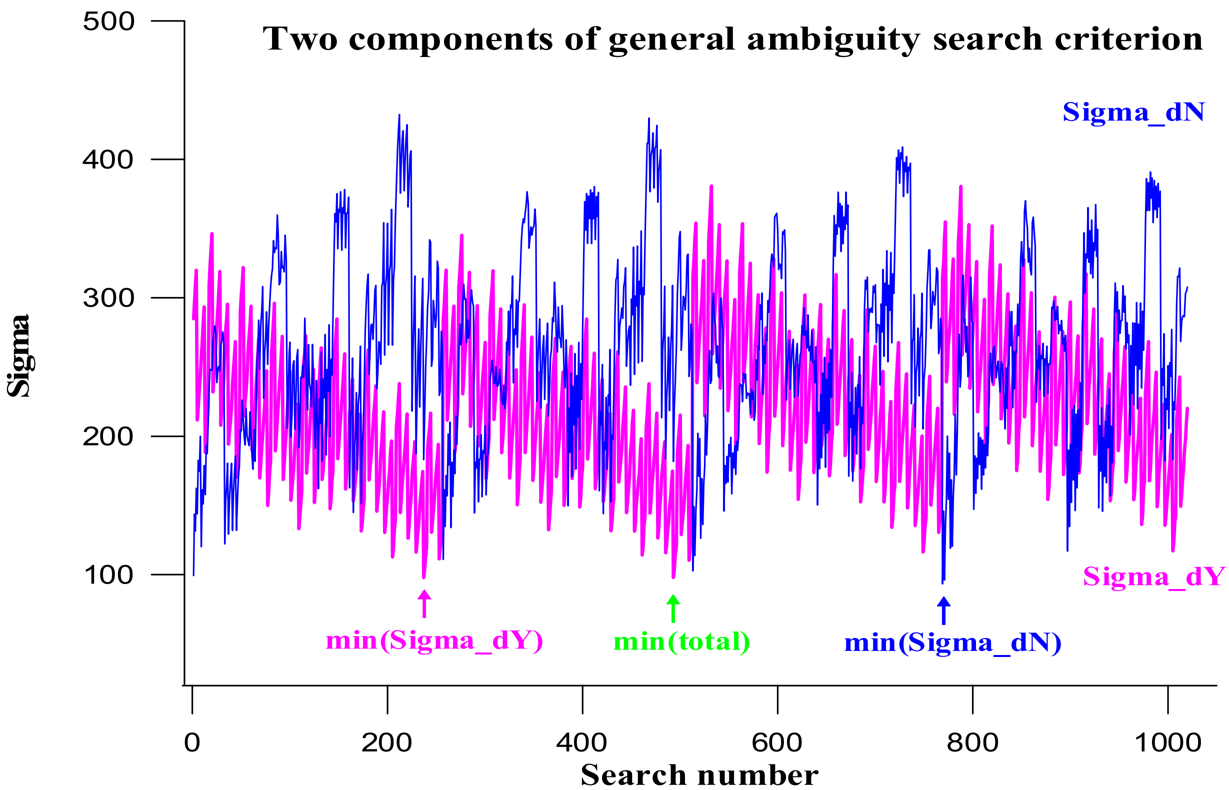

Fig. 1 Two components of the general ambiguity search criterion

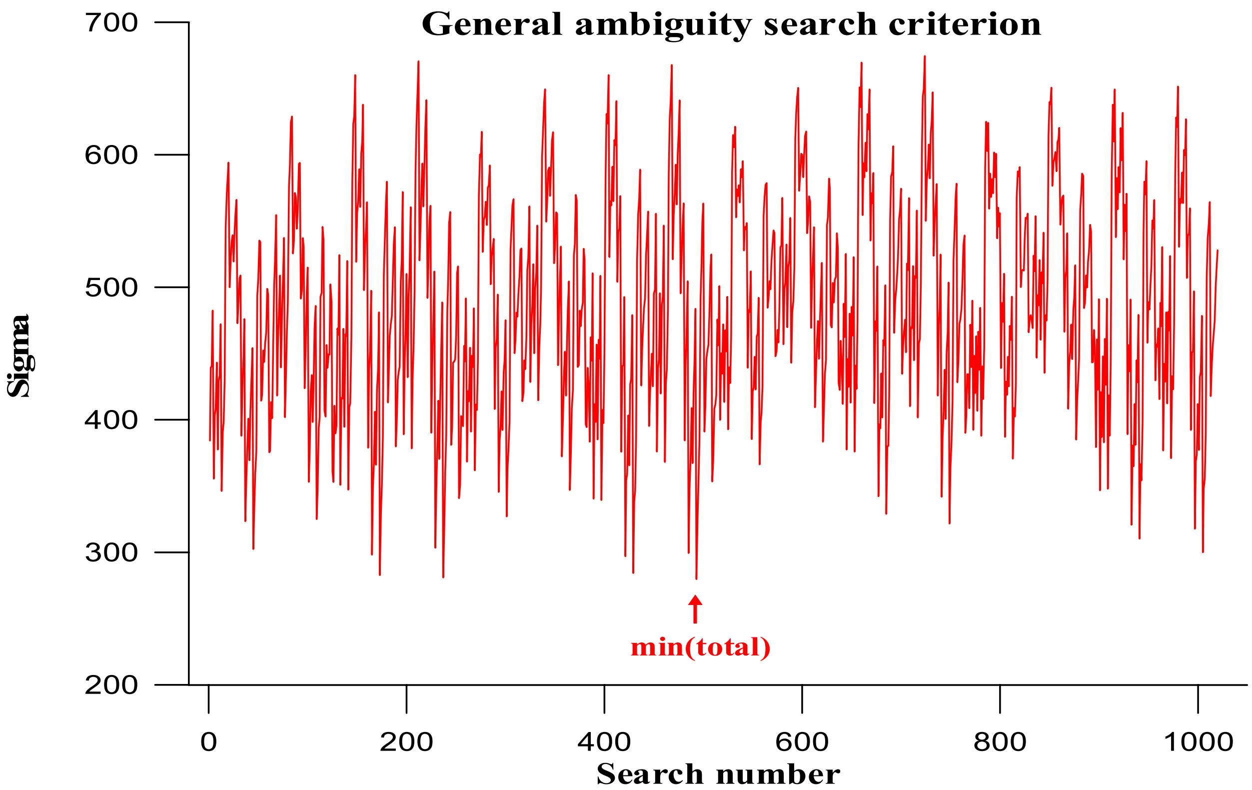

Fig. 2 General ambiguity search criterion

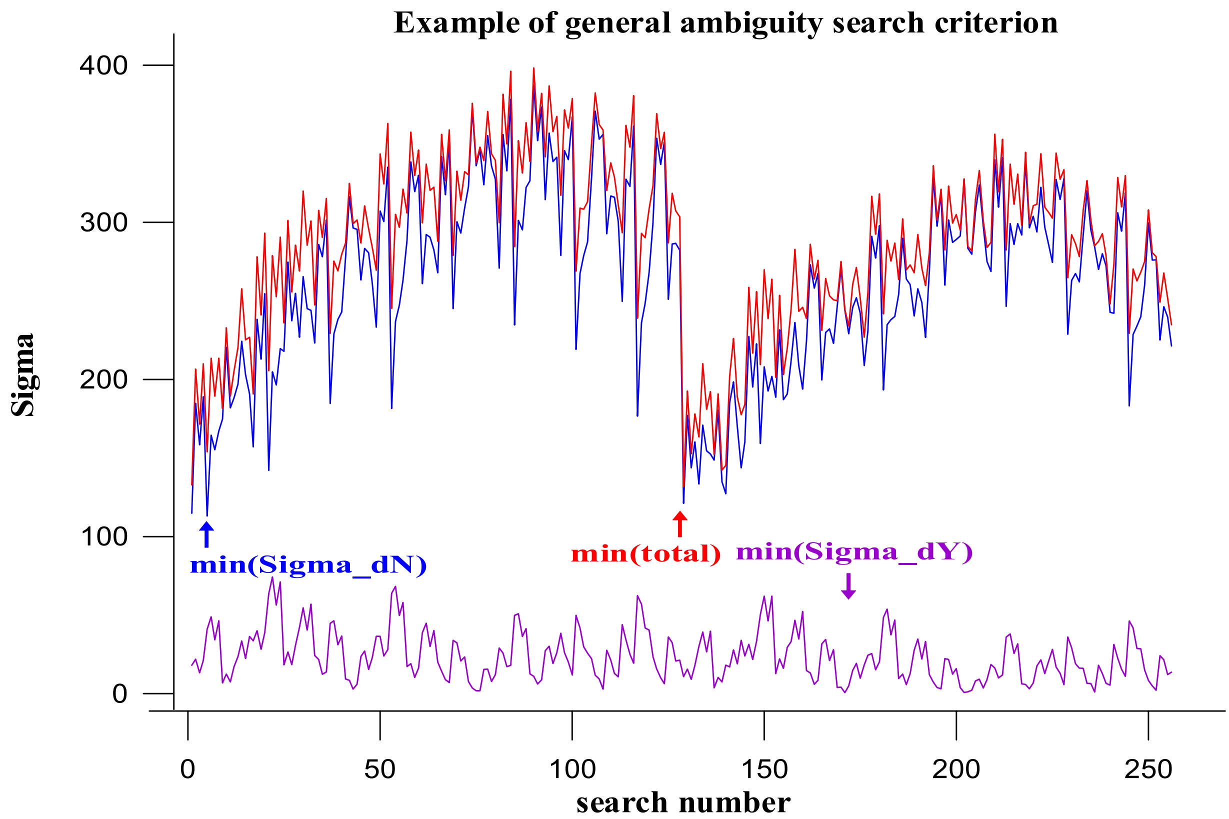

Fig. 3 Example of general ambiguity search criterion

The *δ(dN)* reaches the second minimum at search No. 771. This example shows that the minimum of *δ(dN)* may not lead to the minimum of total sigma, because the related *δ(dY)* is large. If the sigma ratio criterion is used in this case, the LSAS method will reject the found minimum and explain that no significant ambiguity fixing can be made. However, because of the uniqueness principle of the general criterion, the search reaches the total minimum uniquely.

The second example is very similar to the first one. The sigmas of the search process are plotted in Figure 3, where *δ(dY)* is much smaller than *δ(dN)*. *δ(dN)* reaches the minimum at the search number 5 and *δ(dY)* at 171. *δ(total)* reaches the minimum at the search number 129. The total 11 ambiguity parameters are fixed and listed in Table 2. Two ambiguity fixings have just one cycle difference at the 6th ambiguity parameter. The related coordinate solutions after the ambiguity fixings are listed in Table 3. The coordinate differences at component *x* and *z* are about 5 mm. Even the results are very similar, however, two criteria do give different results.

Tab. 2 Two kinds of ambiguity fixing due to two criteria

Ambiguity No.	1	2	3	4	5	6	7	8	9	10	11
LSAS fixing	0	0	1	0	0	0	-1	0	0	-1	-1
General fixing	0	0	1	0	0	-1	-1	0	0	-1	-1

Tab. 3 Ambiguity fixed coordinate solutions (in meter)

Coordinates	x	y	z
LSAS fixng	0.2140	-0.0449	0.1078
General fixing	0.2213	-0.0465	0.1127

Tab. 4 Sigmas of ambiguity search process

Search No.	δ(dN)	δ(dY)	δ(total)
1	248.5681	129.0555	377.6236
2	702.6925	58.9271	761.6195
3	889.5496	107.9330	997.4825
4	452.1952	42.3226	494.5178
5	186.7937	112.3030	299.0967
6	739.0487	55.9744	795.0231
7	931.4125	89.9074	1021.3199
8	592.1887	38.0969	630.2856

In the third example, real GPS data of 3 October 1997 at station Faim (N 38.5295°, E 331.3711°) and Flor (N 39.4493°, E 328.8715°) are used. The sigmas of the search process are listed in Table 4. Both *δ(dN)* and *δ(total)* reach the minimum at the search number 5. This indicates that the LSAS criterion may sometimes reach the same result as that of the equivalent criterion being used.

9 Conclusions and Comments

1). Conclusions

A general criterion of integer ambiguity search is proposed in this paper. The search can be carried out in a coordinate domain, in an ambiguity domain or in both domains. The criterion takes the both coordinate and ambiguity residuals into account. The equivalency of the three searching processes are proved theoretically. The searched result is optimal and unique under the least squares minimum principle and under the condition of integer ambiguities. The criterion has a clear numerical explanation. The theoretical relationship between the general criterion and the common used least squares ambiguity search (LSAS) criterion is derived in detail. It shows that the LSAS criterion is just one of the terms of the equivalent criterion of the general criterion (does not take into account the coordinate change due to the ambiguity fixing). Numerical examples shown that, a minimum δ(dN) may have a relatively large δ(dY), and therefore a minimum δ(dN) may not guarantee a minimum δ(total).

2). Comments

The float solution is the optimal solution of the GPS problem under the least squares minimum principle. Using the general criterion, the searched solution is the optimal solution under the least squares minimum principle and under the condition of integer ambiguities. However, the ambiguity searching criterion is just a statistic criterion. Statistic correctness does not guarantee correctness in all applications. Ambiguity fixing only makes sense when the GPS observables are good enough and the data processing models are accurate enough.

References

Cui, X.; Yu, Z.; Tao, B.; Liu, D. (1982): ***Adjustment in Surveying***, Surveying Publishing House, Peking

Euler, H.J.; Landau, H. (1992): ***Fast GPS ambiguity resolution on-the-fly for real-time applications***. Proceedings of 6th Int. Geod. Symp. on satellite Positioning. Columbus, Ohio, 17-20.

Gotthardt, E. (1978): ***Einfuehrung in die Ausgleichungsrechnung***, Herbert Wichmann Verlag Karlsruhe

Han, S.; Rizos, C. (1995): ***On-The-Fly Ambiguity Resolution for Long Range GPS Kinematic Positioning***, IAG Symposia 115, edited by Beutler, Hein, Melbourne and Seeber.

Han, S.; Rizos, C. (1997): ***Comparing GPS Ambiguity Resolution Techniques***, GPS World, Oct. 1997, 54-61

Hofmann-Wellenhof, B; Lichtenegger, H.; Collins, J. (1997): ***GPS Theory and Practice***, Springer-Verlag, Wien

Leick, A. (1995). ***GPS Satellite Surveying***. New York: John Wiley & Sons.

Remondi, B. (1984): ***Using the Global Positioning System (GPS) phase observable for relative geodesy: modelling***, processing, and results. University of Texas at Austin, Center for Space Research.

Sjoeberg, L. (1998): ***On the estimation of GPS phase ambiguities by triple frequency phase and code data***, ZfV, 123(1998)5, pp. 162-163, Stuttgart, 1998.

Sjoeberg,L. E. (1999): ***Unbiased vs biased estimation of GPS phase ambiguities from dual-frequency code and phase observables***, Journal of Geodesy (1999) 73: 118-124

Teunissen, P.J.G. (1995): ***The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation***. Journal of Geodesy, 70, 65-82.

Wang, G.; Chen, Z.; Chen, W.; Xu, G. (1988): ***The Principle of the GPS Precise Positioning System***, Surveying Publishing House, Peking

Xu, G.; Bastos, L.; Timmen, L. (1997): ***GPS Kinematic Positioning in AGMASCO Campaigns-Strategic Goals and Numerical Results***, Proceedings of ION GPS-97 Conference in Kansas City (USA), p1173-1183

Xu, G.; Schwintzer, P.; Reigber, Ch. (1998): ***KSGSoft (Kinematic/Static GPS Software) --- Software User Manual***, Scientific Technical Report 19/1998, GeoForschungsZentrum Potsdam

Xu, G. (2003): ***GPS - Theory, Algorithms and Applications***, Springer Heidelberg, in English, in print