

# Gaussian Random Process and Its Application for Detecting the Ionospheric Disturbances Using GPS

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**Abstract.** Usually, ionospheric Total Electron Content (TEC) variation with time can be viewed as a stationary random process under quiet conditions. However, sudden events of the Sun and the Earth such as solar flare and sudden commencement of geomagnetic storms may induce the disturbances of the ionosphere, so that the stationary random process is broken; the statistical model parameters change much. Based on this fact, here we make use of the time series of TEC and the auto-covariance function of the stationary process to construct independent identical distribution Gauss sample so that the  $\chi^2$  test can be used to detect the abnormality hidden in the sequence. In addition, GPS data from several IGS sites in China during the severe solar flare occurred on 14th July, 2000 are used to verify the method. The results indicate that the disturbances caused by the solar flare can be effectively detected.

**Key words:** GPS, Total Electron Content, Gaussian Random Process, Ionospheric disturbances

## 1 Introduction

Over the post two decades, GPS has been widely used in studying of the phenomena of Sudden Increase of Total Electron Content (SITEC) caused by the solar flare (Wan. et al., 2000; Zhang et al., 2000 and 2002; Edward et al., 2000), monitoring the large, middle and small scale of the Travelling Ionosphere Disturbance (TID) (Zhang et al., 2002; Ho., et al., 1996; Saito et al., 1998), verifying the theory of Chapman Ionization, analysing the effects of magnetic storm on the ionosphere (Saito et al., 1998; Ho., et al., 1998), monitoring the ionospheric irregularities (Pi

et al., 1997). Due to the high spatial-temporal resolution of Total Electron Content (TEC) data provided by the globally or regionally covered GPS continuously operating reference stations, GPS is helping us further the understanding of the characteristics, the principles and the law of the ionospheric activities at the global, regional or local scale, which greatly promote the development of the high-air atmosphere science and space weather studies. However, most of these researches are accomplished by means of postprocessing, while some special service, such as space weather prediction, wireless communication and high precision GPS geodetic surveying, needs to detect and deal with the disturbing of the ionosphere so that the effects of the ionosphere on them can be best controlled. Therefore, it is necessary that the theory and methods of detecting the ionospheric disturbances using GPS should be studied comprehensively and systematically. As to this subject, Yuan et al. (2001) offered the random ionospheric disturbance detecting theory and scheme for practical operation. The preliminary testing results based on such a theory using the auto-covariance estimation of variable samples (ACEVS) provided by Yuan (et al. 2001) suggest that the ionospheric anomaly can be best detected. Here we will construct the Independent Identical Gauss Distribution (IIDN(0,1)) samples based on the related characteristics of the stationary random process of the variation of the ionosphere, then the hypothesis testing of the chi-square is involved in analyzing the time series of TEC so that the anomaly can be checked out. On the other hand, to validate such a method, the ACEVS method is introduced and the results produced using the two methods are compared.

## 2 Constructing IIDN(0,1) sample with stationary random series

Assume a realization of the ergodic Gaussian stationary random process  $\{x_t\}$  with zero expectation value

$$\tilde{x}_i = x_i + e_i \quad (i = 1, 2, \dots, N) \quad (1)$$

where  $e_i$  is ergodic Gaussian white noise with zero expectation value (independent of  $x$ );  $N$  is the number of samples. For simplicity, the stochastic model and other relevant properties of  $\{x_t\}$  and  $\{e_t\}$  are written as

$$\begin{aligned} E(e_i) &= E(x_i) = 0 \\ COV(x_i, x_{i-r}) &= E(x_i x_{i-r}) = \gamma(r) \\ COV(e_i, e_{i-r}) &= E(e_i e_{i-r}) = \gamma_e(r) \\ COV(e, x) &= E(ex) = 0 \\ COV(\tilde{x}_i, \tilde{x}_{i-r}) &= \gamma_e(r) + \gamma(r) \end{aligned} \quad (2)$$

where  $COV$  is the covariance;  $\gamma$  and  $\gamma_e$  are the autocovariance function of  $\{x_t\}$  and  $\{e_t\}$ , respectively. Since  $e_i$  is ergodic Gaussian white noise with zero expectation value, then the  $\gamma_e$  owns the following characteristics

$$\begin{cases} \gamma_e(y) = 0, & y > 0 \\ \gamma_e(y) = D_e, & y = 0 \end{cases} \quad (3)$$

Therefore the auto-covariance function of sample series  $\{\tilde{x}_i\}$  can be expressed as:

$$\begin{aligned} COV(\tilde{x}_i, \tilde{x}_{i-r}) &= E[(x_i + e)(x_{i-r} + e)] \\ &= \gamma(r) + \gamma_e(r) \\ &= \gamma(r) \quad r > 0 \end{aligned}$$

$$COV(\tilde{x}_i, \tilde{x}_i) = \gamma(0) + D_e \quad r = 0 \quad (4)$$

where  $D_e$  is the variance of the ergodic Gaussian white noise  $e_i$ . Since series  $\{x_t\}$  is an ergodic Gaussian random process, then  $Y = (x_1, x_2, x_3, \dots, x_N)$  can be viewed as  $N$ -variate random vector according to the properties of ergodic random process (Liu, 2000). Here  $E(Y) = 0$ . When variance  $D_e$  of  $e_i$  is known, covariance matrix  $\Sigma_{YY}$  of  $N$ -variate random vector  $Y$  can be determined by using the autocovariance function of series  $\{\tilde{x}_t\}$ . Then vector  $Y$  follows the  $N$ -variate normal distribution with zero expectation and covariance matrix  $\Sigma_{YY}$ . That is:  $Y \sim N(0, \Sigma_{YY})$

where covariance matrix  $\Sigma_{YY}$  is non-negative definite. If  $\det \Sigma_{YY} > 0$ , then random vector

$$Z = \Sigma^{-1/2}(Y - 0) \quad (5)$$

is  $N$ -variate random vector with  $E(Z) = 0$ ,  $\Sigma_{ZZ} = I_N$ , where  $I_N$  is  $n$ -variate unit matrix. Then random vector  $Z$  follows IIDN(0,1).

However, the transferring process described above needs to use the autocovariance function of the stationary random process  $\{x_t\}$ , which can not be accurately known in practice. Then the estimates of autocovariance  $\gamma(r)$  can be derived using samples with the following formula (Peter et al., 1991):

$$\tilde{\gamma}(r) = N^{-1} \sum_{i=1}^{N-r} \tilde{x}_i \tilde{x}_{i-r}, \quad 0 < r \leq N-1 \quad (6)$$

Here Equation (2) is not the unbiased estimates of  $\gamma(r)$ , but in the condition that  $Y = \Sigma^{1/2} \cdot Z$ ,  $\{Z_t\} \sim IID(0, \sigma^2)$ , when  $N \rightarrow \infty$ , the asymptotical distribution  $\tilde{\gamma}(r)$  is  $\gamma(r)$ . Then the estimate  $\tilde{\gamma}(r), r = 0, 1, \dots, N-1$  owns the autocovariance matrix

$$\Sigma_n = \begin{bmatrix} \tilde{\gamma}(0) & \tilde{\gamma}(1) & \dots & \tilde{\gamma}(n-1) \\ \tilde{\gamma}(1) & \tilde{\gamma}(0) & \dots & \tilde{\gamma}(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}(n-1) & \tilde{\gamma}(n-2) & \dots & \tilde{\gamma}(0) \end{bmatrix} \quad (7)$$

Which is non-negative definite when  $n \geq 1$ .

As discussed above, the discrete series of Gaussian stationary random process can be transferred to a multivariate random vector, then the analysis of the statistical characteristic parameters of stationary random process can be substituted by the corresponding analysis of a multivariate random vector. Therefore, when it happens to be anomaly at the  $i$ th observation of  $\{x_t\}$ , the corresponding  $i$ th of  $\{Z_t\}$  should be anomaly. Then the statistical tools can be used to analyze it.

## 3 Determining stationary Ionospheric TEC series with GPS

The GPS TEC observations include the deterministic  $I$  part (such as a trend and a period) and stochastic ( $\delta I$ ) part due to the ionosphere activity. Usually, for short time scales, for deterministic effects, only the trend variation

can be considered, and  $I$  can be written as a polynomial

$$I_t = \sum_{i=0}^m a_i t^i$$

The stochastic effects  $\delta I$  may be considered as a Gaussian random process with zero expectation value. When random disturbances of the ionosphere happen, their effects on TEC will usually destroy the steady state of  $\delta I$ . Therefore, it is possible to test the change of state of  $\delta I$  using statistical tools.

Assume that  $\tilde{I}_t$  is the ionosphere TEC observation at an arbitrary epoch  $t$  and  $\varepsilon_i$  is its Gaussian white noise  $\{E(\varepsilon_i) = 0\}$ , independent of  $\delta I$  [i.e.  $E(\delta I_t \varepsilon_{t+i}) = 0$ ]. Further,  $\{\delta I + \varepsilon\}$  is a Gaussian random stationary process with zero expectation value. Thus the ionospheric TEC observation model can be expressed as

$$\tilde{I}_t = I_t + \delta I_t + \varepsilon_i = \sum_{i=0}^m a_i t^i + \delta I_t + \varepsilon_i \quad (8)$$

Define the difference operator as

$$\begin{aligned} \nabla \tilde{I}_t &= \tilde{I}_{t+1} - \tilde{I}_t \\ \nabla^k \tilde{I}_t &= \nabla(\nabla^{k-1} \tilde{I}_t) = \sum_{i=0}^k (-1)^i C_k^i \tilde{I}_{t+k-1} \end{aligned} \quad (9)$$

where  $C_k^i$  is the combination operator.

To reduce the trend term  $I_t$ , a  $q = m + 1$ -order difference operation can be used for Eq. (8)

$$\begin{aligned} \nabla^q \tilde{I}_t &= \nabla^q \delta I_t + \nabla^q \varepsilon_i = \nabla^q (\delta I_t + \varepsilon_i) \\ E(\nabla^q \tilde{I}_t) &= \nabla^q E(\delta I_t + \varepsilon_i) \\ &= \nabla^q E(\delta I_t) + \nabla^q E(\varepsilon_i) = 0 \end{aligned} \quad (10)$$

Similarly

$$\begin{aligned} \nabla^q \tilde{I}_{t+h} &= \nabla^q \delta I_{t+h} + \nabla^q \varepsilon_{i+h} \\ E(\nabla^q \tilde{I}_{t+h}) &= 0 \end{aligned} \quad (11)$$

From the above, it can be seen that  $\nabla^q \tilde{I}_t$  is a linear combination of  $\delta I_{t+q-i} + \varepsilon_{i+q-i}$ , ( $i = 0, 1, 2, \dots, q$ ), while  $\{\delta I_{t+q-i} + \varepsilon_{i+q-i}\}$  is a Gaussian random variable with zero expectation value. So according to the invariance property of linear transformations of Gaussian distributions,  $\nabla^q \tilde{I}_t$  is a Gaussian random variable with zero expectation value as well. Then it can be proved that

$\{\nabla^q \tilde{I}_t\}$  is stationary and obviously is an ergodic process as well [Yuan, et al., 2001].

Because  $\{\nabla^q \tilde{I}_t\}$  is an ergodic Gaussian process with zero expectation value, and if  $\tilde{x}_t = \nabla^q \tilde{I}_{t+h,t}$  and  $x_t = \nabla^q I_{t+h,t}$ , then, under normal observation conditions, the series  $\{\tilde{x}_t = \nabla^q \tilde{I}_{t+h,t}\}$  may be considered

as the approximate series of  $\{x_t = \nabla^q I_{t+h,t}\}$  and can be transformed to IIDN(0,1) samples according to the method discussed in Sect. 2. In the time series of TEC observations of GPS, the change of the statistical properties of the random ionospheric TEC  $\{x_t = \nabla^q I_{t+h,t}\}$  from a status of stability to one of disturbance can be distinguished by the change of its transformed IIDN(0,1) samples. So it is possible to test by using the GPS time series.

## 4 Application and analysis

### 4.1 Scheme for detecting the Anomaly

According to what have been discussed above, we can construct the scheme for testing the anomaly using the transformed IIDN(0,1) samples. Here we briefly describe the scheme as following:

1) Get the differenced TEC series from GPS observations so that it is stationary. Usually, second-order differencing is enough.

2) Calculate the estimates of the auto-covariance of the differenced stationary TEC series using formula (6), construct the auto-covariance matrix with formula (7). Here we set 40-order matrix so that the fixed-length samples can be sliding along the time series with time.

3) According to the fixed-length samples windows (40 samples used here) sliding with time, construct the IIDN(0,1) samples  $\{Z_t, t = 1, 2, \dots, N\}$  using formula (5). Then  $\chi^2$  hypothesis testing can be used to detect the anomaly.

4) If  $\chi^2(N) = Z_1^2 + Z_2^2 + \dots + Z_N^2 \leq \chi_{\alpha}^2(N)$ , then the state of ionosphere is stable, otherwise an anomaly happens.

To validate the scheme above, here we also apply the ACEVS method to the sample example. The core principle of ACEVS is to construct an asymptotically independent normal Gaussian sequence  $\{\rho_{\nabla}(r) \sim N(0,1)\}_M^N$  so that  $\chi^2$  hypothesis testing can be

used. Here  $M$  is the minimum samples used to get  $\tilde{\gamma}(r)$  using formula (6),

$$\rho_{\nabla}(r) = \frac{\Delta \tilde{\gamma}_{N+1,N}(r)}{\sqrt{\tilde{\gamma}^2(0)/N(N+1)}}$$

$$\Delta \tilde{\gamma}_{N+1,N}(r) = \tilde{\gamma}_{N+1}(r) - \tilde{\gamma}_N(r). \quad (12)$$

To simplify the ACEVS method in application, we briefly describe the scheme for testing an anomaly using ACEVS method as following:

1) Get the differenced TEC series from GPS observations so that it is stationary. Usually, second-order differencing is enough.

2) According to the  $N$  samples we obtained (here we set  $r=10$ ,  $M=100$ ,  $N>100$ ), calculate  $\gamma_N(r)$  and construct sequence  $\left\{ \tilde{\gamma}_i(r), i = 1, 2, \dots, N - M \right\}$ .

3) Get the series

$$\left\{ \Delta \tilde{\gamma}_{i,i+1}(r) = \tilde{\gamma}_{i+1}(r) - \tilde{\gamma}_i(r), i = 1, 2, \dots, N - M + 1 \right\},$$

then construct new sequence

$$\left\{ \rho_{\nabla}(r) = \Delta \tilde{\gamma}_{i,i+1}(r) / \sqrt{\frac{\tilde{\gamma}^2(0)}{N(N+1)}}, i = 1, 2, \dots, N - M + 1 \right\}.$$

This sequence is an asymptotically independent normal Gaussian sequence.

4) Using the fixed-length sliding sample window (here we set it as 40, so  $N-M$  must be more than 40), constructs

the statistical quantity  $\chi^2(k) = \sum_{i=1}^k \rho_{\nabla}^2(r)$ . Then slides the

window with time and test the status of ionosphere, if  $\chi^2(k) \leq \chi_{\alpha}^2(k)$ , the ionosphere is in good condition.

### 4.2 Example and analysis

The increase and decrease of the ionospheric TEC with time are the main phenomena that reflect the status of the ionosphere. Usually, solar flare and the irregular activities of the atmosphere may cause the phenomena of SITEC so that the quiet status of the ionosphere is broken. Here we will analyze the anomaly of the ionosphere caused by the strong solar flare happened on 14<sup>th</sup>, July, 2000. The two testing schemes described above are used in the following section.

Fig. 1 provides the first-order differenced TEC curves derived from the observations of GPS satellites PRN29 and PRN21 observed at IGS sites WUHN and BJFS respectively on 14<sup>th</sup>, July, 2000. Obviously, solar flare caused the phenomena of SITEC during the period of UTC10:00~10:30. The amplitude of such sudden increase was up to 0.76TECU/min and lasted almost half an hour. Such ionospheric anomaly could be seen in the TEC observations of other satellites observed at other GPS tracking sites, which means that the solar flare affected the ionosphere in the global scale.

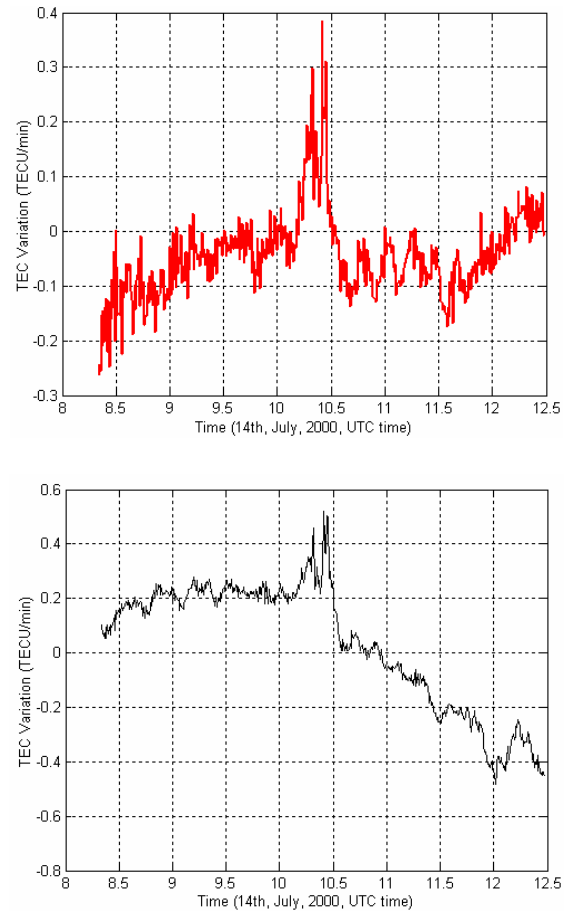


Figure 1. The first-order differenced TEC series observed by PRN29 (up panel) and PRN21(down panel) at WUHN and BJFS respectively on 14<sup>th</sup>, July, 2000

In Fig. 1 we can see that the trend of the first-order differenced TEC curves is obvious, which indicates that the series are not an stationary process. So the second-order differenced TEC series of PRN29 observed at WUHN site is derived, which is shown in Fig. 2(a). Fig. 2(b) provides the value of the auto-correlation function of  $R(10)$  varying with time (the anomaly samples are kicked out), which is a constant and indicates that it is independent of time. Here  $R(10)$  is as a example, other values, which is similar to that of Fig. 2(b), of the auto-correlation function varying with time at different

intervals have been derived and not shown here. So the second-order differenced TEC series is stationary.

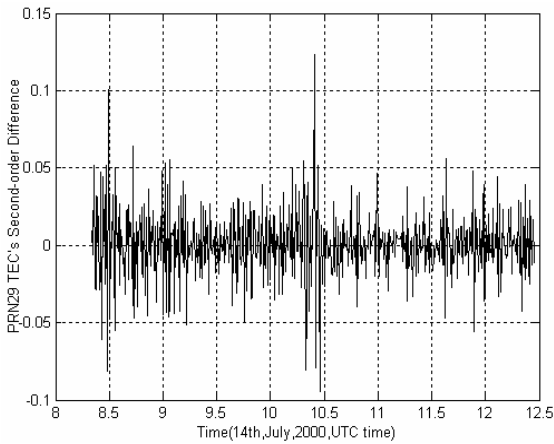


Figure 2(a). The second-order differenced TEC series observed by PRN29 at WUHN

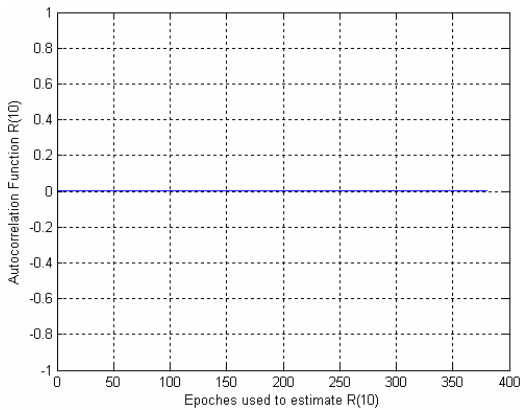


Figure 2(b). The Auto-correlation Function of R(10) (the interval is 10 epochs) of the second-ordered differenced TEC series

Fig. 3 is what's obtained by the  $\chi^2$  hypothesis test using the IIDN(0,1) samples transformed from the second-order differenced TEC series with the approximate autocorrelation matrix derived from such a random process, while Fig. 4 is the result of the  $\chi^2$  hypothesis test using ACEVS method with the same realization of such a random process. In Fig. 3 and Fig. 4, we can see that the two figures are in the similar shape, what's more, the peaks appears and corresponds to the anomaly period reflected in Fig. 1. In addition, the value of  $\chi^2$  is more than the threshold value 20.72 before the peak value appear, which indicate that the ionosphere begun to be unstable before the solar flare break out, while the ionosphere recovered quiet soon after the solar flare. This reflects the characteristic of the events of that solar flare on the ionosphere. The similarity of such curves in Fig. 3 and Fig.4 show that the two schemes described above can

achieve the same purpose of monitoring and detecting the anomaly of the ionosphere.

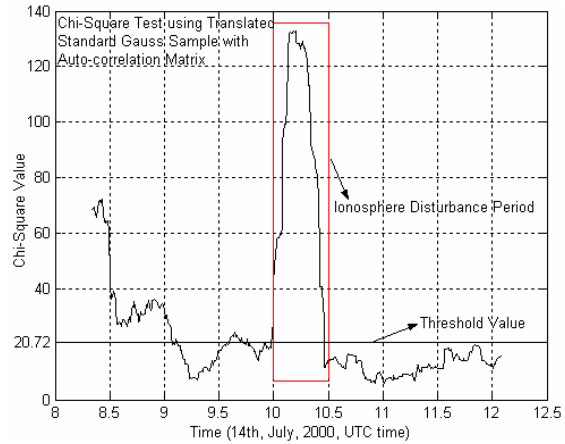


Figure 3. The result of  $\chi^2$  testing with the transformed IIDN(0,1) Samples

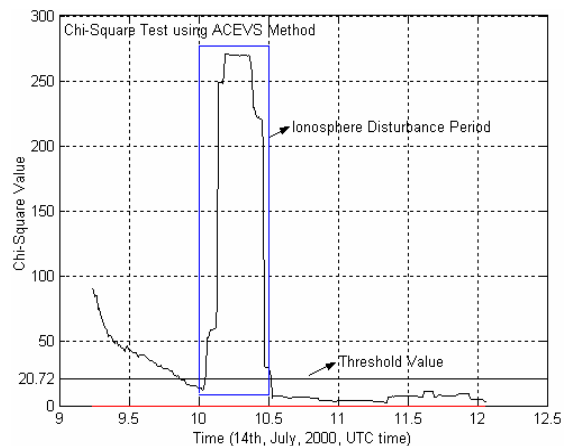


Figure 4. The result of  $\chi^2$  hypothesis test using the ACEVS method

#### 4. Summary

The breakage of the quiet ionosphere corresponds to the change of the statistical parameters of the TEC time series, which can be used to monitor the activities of the ionosphere so that the disturbance of the ionosphere can be detected. Monitoring and detecting the ionospheric disturbance is important for the research and prediction of the space weather, as well as GPS surveying, satellite navigation, satellite communication and so on. So the theory and methods of real-time monitoring of the ionosphere with GPS can lead us to know the status of the ionosphere accurately and in real time, so that we can take good steps to avoid great loss. Therefore, the methods used to detect the disturbance of the ionosphere

with the stationary random process in this paper provide a good alternative choice.

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