Real-time Doppler/Doppler Rate Derivation for Dynamic Applications

Jason Zhang

School of Mathematical and Geospatial Sciences, RMIT University
e-mail: jzhang@eos-aus.com Tel: + 61-02-6222 7949

Kefei Zhang, Ron Grenfell, Yong Li and Rod Deakin

School of Mathematical and Geospatial Sciences RMIT University

Abstract. Precise GPS velocity and acceleration determination relies on Doppler and/or Doppler rate observations. There are no direct Doppler rate measurements in GPS. Although every GPS receiver measures Doppler shifts, some receivers output only "raw" Doppler shift measurements and some don't output any at all. In the absence of raw Doppler and Doppler rate measurements, a differentiator is necessary to derive them from other GPS measurements such as the carrier phase observations. For real-time dynamic applications, an ideal differentiator should have a wideband frequency response to cover all the dynamics. It should also have a group delay as short as possible. In addition, a low-order differentiator is more favourable for easy implementation.

This paper provides an overview of methods in differentiator design for applications of GPS velocity and acceleration determination. Low-order Finite Impulse Response (FIR) differentiators proposed by Kavanagh are introduced. A class of first-order Infinite Impulse Response (IIR) differentiators are developed on the basis of Al-Alaoui's novel differentiator. For noise attenuation, it is proposed to selectively use Kavangagh's FIR differentiators, and the first-order IIR filters derived for adaptation to different dynamics.

Key words: GPS velocity determination, GPS acceleration determination, differentiator design, FIR filter, IIR filter, Doppler.

1. INTRODUCTION

Previously proposed methods for GPS velocity and acceleration determination fall in two categories. One is to derive velocity and acceleration directly from GPS determined positions, another is based on the Doppler shift method. The latter has several advantages: it doesn't rely on the precision of the positions from GPS, nor will the accuracy dramatically degrade with an increase in sampling rate (say 10Hz or more). Since there is no direct Doppler rate observation in GPS measurements, as a "virtual" observable, it must be derived in order for the formulae presented by Jekeli and Garcia (1997) to be applied directly in the Doppler shift method.

Every GPS receiver measures Doppler shifts. However, this is primarily an intermediate process to obtain accurate carrier phase measurements. Thus the quality of Doppler shift output varies from receiver to receiver depending on manufacturer. The Trimble 5700™ geodetic receiver, for instance, has a measurement precision of ±1mm/s. The observed Doppler is from a tracking loop that is updated at a very high rate. This also enables the receiver to sense phase accelerations (Harvey 2004). Unfortunately the sensed phase acc-elerations and the Doppler shift on L2 are discarded. Some other GPS receivers, for example the Superstar II™ from NovAtel, have only code and L1 phase outputs (SuperstarII 2004) and the Doppler shifts are masked out of the measurements. For our purposes to obtain accurate velocity and acceleration using these types of receivers, it is necessary to derive the Doppler shifts, i.e. the change rates of the carrier phase from the measured carrier phase measurements.

Differentiators are required to get the Doppler rate "observable" for any type of receiver, or to get the Doppler shift from the carrier phase. In real-time and dynamic applications it is also desirable that the designed

differentiator should have a wideband frequency response to cover the system dynamics. It should also have a group delay as short as possible so as to get the Doppler shift or Doppler rate instantaneously. For those receivers that output only "raw" Doppler shifts, the derivation of precise Doppler from the carrier phase plays a key role in precise velocity and acceleration determination. This is because the precision of carrier phase observables can be fully exploited. The objective of this paper is to explore the techniques to derive Doppler rate from GPS measurements, or to derive precise Doppler shift from the carrier phase in real time and in dynamic situations.

Several investigations have been conducted for this purpose in the GPS measurement domain, and the proposed methods can be categorised into:

(1) Curve fitting (Fenton and Townsend, 1994);

(2) Kalman smoother/filtering (Hebert, Keith et al. 1997);

(3) Taylor series approximation (Hebert, Keith et al. 1997; Cannon, Lachapelle et al. 1998; Bruton, Glennie et al. 1999);

(4) Finite Impulse Filter (FIR) by using Fourier series with window techniques (Bruton, Glennie et al. 1999); and

(5) FIR optimal design using the Remez exchange algorithm (ibid).

The FIR filtering technique based on Taylor series approximations was recently adopted to derive phase accelerations by Kennedy (2003).

This paper briefly describes the digital differentiator theory and states the design problems in real-time dynamic GPS applications. It is followed by a comprehensive literature review on each method referred to in the above section. By comparing the various differentiator designs, a series of first-order Infinite Impulse Filters (IIR) are presented which are capable of delivering the derivatives from input signals in real-time dynamic situations. An adaptive scheme is also proposed for noise attenuation.

2. Digital filter and digital differentiator Design

2.1. Digital Filtering

Suppose there is a discrete signal sequence of x_n (n is an integer) with a sampling period of T. A digital filter can be regarded as a linear combination of the discrete samples x_{n-k}, together with the previous output y_{n-k}, which can be defined by the following formula (Hamming 1977, p2):

$$y_n = \sum_{k=-\infty}^{k=-\infty} c_k x_{n-k} + \sum_{m=1}^{M} d_k y_{n-m} \qquad (1)$$

where y_n is the output of the filter, c_k and d_k are filter coefficients which are referred to as the impulse response of the filter, which is the filter response for a unit input signal pulse (ibid).

The coefficients of a filter completely define the property of the filter and selectively suppress or enhance particular parts of signals. When the coefficients of the second term on the right hand side of Eq. (1) are nonzero, the filter is referred to as a *recursive* filter since the output of y_{n-k} has been used recursively. The filter coefficients c_k and d_k are usually time-invariant in classical filter designs. Their values are carefully chosen to achieve the desired filtering result. However, their values can be assigned on-line to respond to the change of situations in the so-called adaptive filter design. For practical applications, the length of a realisable digital filter is always finite.

2.1.1. Transfer function

The transfer function of a discrete filter is defined as the Z-transform of the filter output signal over the Z-transform of the input signal, i.e.

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{\sum_{k=-\infty}^{k=\infty} c_k \cdot Z(x_k)}{1 - \sum_{m=1}^{M} d_m \cdot Z(y_m)} = \frac{\sum_{k=-\infty}^{k=\infty} c_k \cdot z^{-k}}{1 - \sum_{n=1}^{M} d_k \cdot z^{-n}} \qquad (2)$$

where z is a complex variable, and the Z-transform is a linear transform whereby a discrete-time signal value of x_n is defined as

$$Z(x_{n-k}) = X_{n-k}(z) = z^{-k} x_n \qquad (3)$$

and where z^{-1} serves as a unit delay operator. The transfer function is most important in filter design and analysis. With the transfer function having been determined, one can directly write out the impulse response of the filter (filter coefficients), and further analyse the performance of the filter either in the time domain or in the frequency domain.

2.1.2. Frequency and amplitude response

The frequency response of a filter is defined as the discrete Fourier transform of output signals over the discrete Fourier transform of input signals

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum\limits_{k=-\infty}^{k=\infty} c_k \cdot e^{-j\omega k}}{1 - \sum\limits_{m=1}^{M} d_m \cdot e^{-j\omega m}} \qquad (4)$$

The above frequency response function is obtained by simply replacing the variable z in the transfer function by the Fourier transform variable e^{jw}. The frequency response function allows us to evaluate the frequency response of a filter on the unit cycle.

Factoring the magnitude of the frequency response into the following form

$$\|H(\omega)\| = G(\omega) \cdot e^{j\Theta(\omega)} \qquad (5)$$

gives the amplitude response $G(\omega)$ which is the gain of the filter. The phase response $\Theta(\omega)$ shows the radian phase shift experienced by each sinusoidal component of the input signal. The phase and group delays of a filter give the time delay in seconds experienced by each sinusoidal component of the input signals:

$$phase \quad delay \equiv \frac{\Theta(\omega)}{\omega} \qquad group \quad delay \equiv \frac{d}{d\omega}\Theta(\omega) \qquad (6)$$

In the case of a filter that has a linear phase response, the group delay and the phase delay are identical, for example when $\Theta(\omega) = \frac{\pi}{2} \cdot \omega$.

2.1.3. Noise amplification

A digital filter is a linear combination of input signals that are usually contaminated by noise. For simplicity we assume that the noise is Gaussian white, and thus the error propagation law applies. This allows us to estimate the noise amplification of the filter. Assume that the noise of a series of L1 carrier phase measurements $x_n = x_n^0 + \varepsilon_n$ is Gaussian white, where x_n^0 stands for the true value of x_n, and then the outcome of the finite non-recursive filter is

$$y_n = \sum_{k=-K}^{k=K} c_k x_{n-k} = \sum_{k=-K}^{k=K} c_k x_{n-k}^0 + \sum_{k=-K}^{k=K} c_k \varepsilon_{n-k} \qquad (7)$$

and the variance of the filter can be evaluated by (Hamming 1977, p14)

$$E\left\{ \sum_{k=-K}^{k=K} c_k \varepsilon_{n-k} \right\}^2 = \sum_{k=-K}^{k=K} c_k^2 E[\varepsilon_{n-k}]^2 = \sigma_x^2 \sum_{k=-K}^{k=K} c_k^2 \qquad (8)$$

This shows that the sum of the squares of each coefficient of a filter determines the noise amplification of the filtering process.

Supposing that the variance of a recursive filter is $\sigma_{y_n}^2$, and applying the preceding procedures, we have

$$\sigma_{y_n}^2 = \sigma_x^2 \sum_{k=-\infty}^{k=\infty} c_k^2 + \sum_{m=1}^{M} d_{n-k}^2 \sigma_{y_{n-m}}^2 \qquad (9)$$

Let us further assume that $\sigma_{y_{n-1}}^2 = \sigma_{y_{n-2}}^2 \cdots = \sigma_{y_{n-M}}^2 = \sigma_{y_n}^2$, and then the variance of the filter can be estimated by

$$\sigma_{y_n}^2 \approx \frac{\sum\limits_{k=-\infty}^{k=\infty} c_k^2}{1 - \sum\limits_{m=1}^{M} d_{n-m}^2} \cdot \sigma_x^2 \qquad (10)$$

This indicates that we can either roughly estimate the variance of the recursive filter or "precisely" calculate the filter variance by computing the initial variance of the recursive filter using Eq. (10), and then estimating the variance of the filtered signals using Eq. (9).

2.2. Statement of Problem of Differentiator Design

Differentiator design has been the subject of extensive investigation in digital signal processing. A main issue is that a differentiator amplifies noise at high-frequencies (Carlsson, Ahlen et al. 1991). As GPS signals are of low

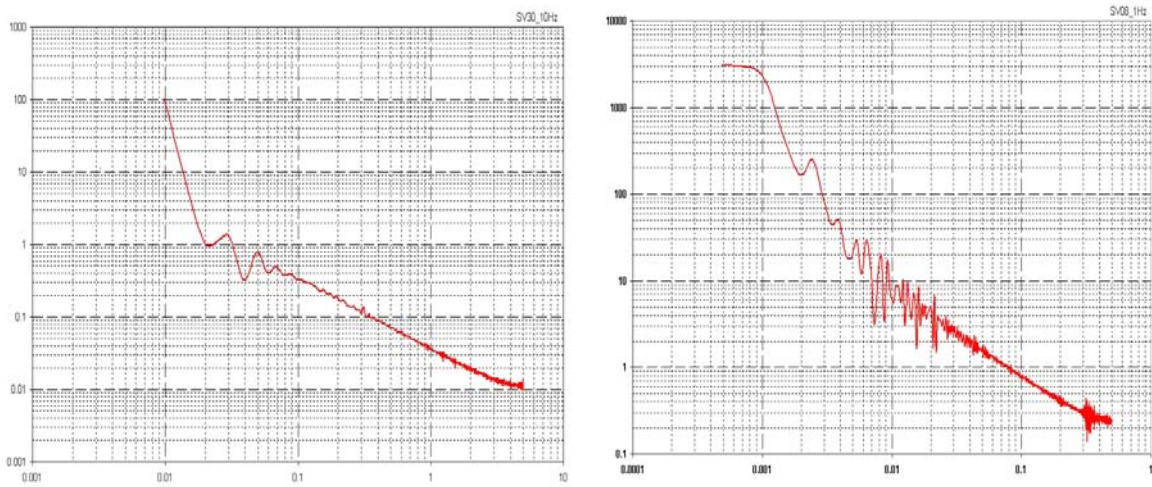*Journal of Global Positioning Systems*

Fig. 1: Power spectral densities for the 1Hz and 10Hz carrier phase signals

frequency character, (see Fig. 1), it is suggested that a low pass filter would be suitable for the design of differentiators. However, the change of dynamics in a system is normally of high frequency. Hence we have to deal with the complicated high frequencies with a broad/full band differentiator. Another complication arises from the signal correlation. It is shown that the GPS carrier signals can be regarded as Gaussian white only when the sampling rate is lower than 1Hz; when the sampling rate goes higher, time correlations must be considered (Bona 2000; Borre and Tiberius 2000). Thirdly, the differentiation may be affected due to lack of information on future signals since the application is real time oriented. Finally there might be aliasing problems due to sampling.

So the problem is to get the derivative from GPS observations where both the signals and noise have random characteristics. In the case of corrupting noise being wideband white and the signal being a Gauss-Markov process (mostlikely for GPS applications), it is apparent that no differentiator is going to be perfect in passing the desired derivative whilst suppressing the noise (Brown and Hwang 1992, p172). This is a typical Wiener filter problem (ibid). The solution is a compromise between good differentiation and low noise sensitivity to achieve a small total error.

The Kalman filter is a space-state solution of the Wiener filter problem (ibid), which is formulated by using the minimum mean-square-error estimation criterion in a two-step recursive procedure. By assuming that both the process driving noise and the measurement noise are Gaussian white and there is no correlation between them, it first predicts the signal state using the system dynamic equation, and then updates the prediction with measurements to get estimates. A successful Kalman filter is subject to proper modelling of system dynamics and the associated stochastic random process. It is suggested that the less than satisfactory performance of

the Kalman filter in the case of Heber et al. (1997) is not due to the Kalman filter approach itself, but due to the improper modelling of the system state when it is highly dynamic.

When the sampling rate is high, the theoretical difficulties in Kalman filtering are mainly in the determination of the random process of system driving noise, and the handling of correlations of measurement noise and the cross-correlation between the measurement and signal noises. Another associated practical problem is the heavy computational load in real-time data processing. Finally the outcome of a Kalman filter is a smooth, band-limited solution (Bruton, Glennie et al. 1999). Therefore, it is reasonable to find solutions in the frequency domain rather than in the state space using Kalman filters.

The digital differentiator design oriented in the frequency domain should still consider the variance of the output. Thus the criteria of the differentiator may be summarised as follows:

- the magnitude of frequency response is accurate in low frequencies and is as close to the ideal differentiator $H(\omega) = j\omega$ (Stearns 2003, p127) as possible in a broad band sense depending on the system dynamics;

- the phase response is linear or approximately linear;

- the group delay is acceptably small;

- the sum of the squares of filter coefficients can be minimized; and

- easy to be implemented in real time , i.e. to be causal and low order since there are cycle slips and loss-of-lock of signals.

3. Taylor series approximations

Taylor series approximations have been widely used to derive differentiators. The differentiators used by (Cannon, Lachapelle et al. 1998), Hebert (1997) and Kennedy (2002; 2003) are of low order Taylor series. They are all in the form of central difference approximations such as

$$y_n = \sum_{k=-N}^{N} c_k \cdot x_k \qquad (11)$$

where N is the order of Taylor series approximation. Fig. 2 depicts the frequency responses of some low order central difference Taylor series approximations.
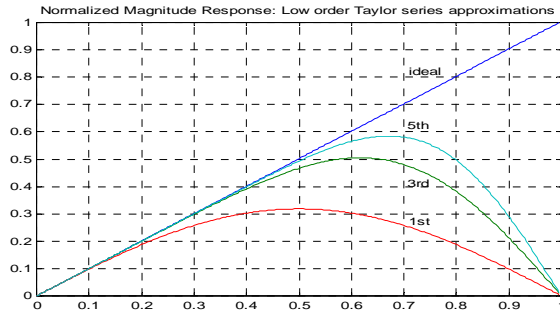
Fig. 2: Frequency response of low order central difference Taylor series approximation

It is apparent that the higher the order, the closer that a Taylor series approximation is to the ideal differentiator. This suggests that broad band differentiators can be designed based on Taylor series, and this can be observed in Khan and Ohba (1999), who gave the explicit coefficients c_k by

$$c_0 = 0$$

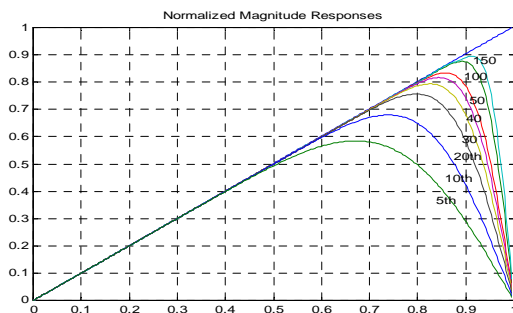$$c_k = c_{-k} = \frac{(-1)^{k+1}N!^2}{k(N-k)!(N+k)!} \qquad (12)$$

Fig.3: Frequency responses for arbitrary order Taylor series approximations

As can be seen from Fig. 3, this type of differentiator is characterised as having zero amplitude response in both $\omega=0$ and $\omega=1$ (Nyquist frequency). Actually this is the property of type III FIR filters (Chen 2001,p299) which will be discussed later.

4. Curve fitting with window

Jekeli and Garcia (1997) used fifth-order B-splines to derive phase accelerations, and Fenton and Townsend (1994) adopted parabolic functions to obtain the precise Doppler. The referenced curve fitting techniques use sliding windows wherein the data are fitted into polynomials using the least squares approach. The derivative of the central point of a window is obtained by differentiating the polynomials with respect to time accordingly.

Bruton (1999) gave an in-depth review of the curve fitting differentiators. It is concluded that whether a curve fitting uses a polynomial, a parabola, or a cubic spline, the resultant differentiator approaches the ideal only at lower frequencies. Since it is band-limited and lowpass, it is suitable only for low dynamic or static applications. Furthermore, performing the least squares estimation involves intensive computation. Moreover, to obtain the current derivative at t_0, the curve fitting with window requires the input at t_k, which is a signal in the future. Therefore we may conclude that the windowed curve fitting approach is inappropriate for real-time dynamic applications.

5. FIR filters

A Finite Impulse Response (FIR) filter consists of a series of multiplications followed by a summation. The FIR filter operation can be represented by the following equation (Hamming 1977)

$$y_n = \sum_{i=-K}^{K} c_i \cdot x_{n-i} \qquad (13)$$

A filter in this form is named FIR because the response to an impulse dies away in a finite number of samples. Note that this form is non-causal and unrealisable. In order to present a causal FIR differentiator, changing the form is required. This leads to

$$y_n = \sum_{i=0}^{N} c_i \cdot x_{n-N} \qquad (14)$$

The Fourier series with window are classical in the design of FIR filters where the impulse response is calculated by the inverse discrete Fourier transform of the transfer function, i.e. (Chen 2001)

$$c_d[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\omega=\pi} j\omega \cdot e^{-j(n-M)\omega} \cdot d\omega$$

$$= \frac{\cos[(n-M)\pi]}{(n-M)} - \frac{\sin[(n-M)\pi]}{\pi(n-M)^2} \qquad (15)$$

where M=N/2 and the infinite length of Fourier terms is truncated into finite terms. The truncation may cause a discontinuity at the edges of the window and leads to residual oscillations named Gibbs oscillations (ripples in the amplitude response against frequency). Different window methods can be used to smooth the glitches, truncate the filter coefficients, and sharpen the frequency response. Fig.4 gives the comparison between direct truncation and applying the Kaiser window technique.

5.1. Type III FIR Differentiator Design

A FIR filter of type III has an odd length and anti-symmetric impulse response. In this case, the differentiator's coefficients are

$$c_d(n) = \begin{cases} \dfrac{\cos[(n-M)\pi]}{(n-M)} & \text{for} \quad n \neq M \\ 0 & \text{for} \quad n = M \end{cases} \qquad (16)$$

where the *sine* term in Eq. (15) vanishes. To eliminate the Gibbs phenomenon due to the finite truncation, a window function is required. Among many windows that are available, the Kaiser window is most popular. It can be evaluated to any desired degree of accuracy using the rapidly converging series of the zero-order Bessel function of the first kind (Farlex 2004). The ripple of the stopband can also be controlled by an adjustable variable α to meet the optimal criteria given by Kumar and Roy (1988) and Selesnick (2002). With the above procedures, one can also design FIR differentiators with different cut-off frequencies.
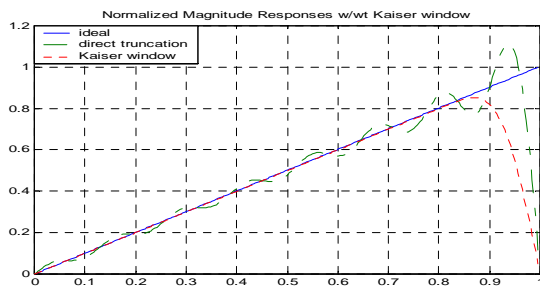

Fig. 4: Magnitude responses of FIR differentiators based on the window technique

Theoretically, FIR filters of type III can be designed to meet requirements at nearly all frequencies, as long as we increase the filter order. However, since the frequency response to the Nyquist frequency is zero, it is impossible to design a full band type III differentiator. Although such filters are causal and are linear in phase, the actual derivative obtained is with respect to time *t-(N/2)T*. This means that the more taps in a FIR filter, the longer the group delay will be. This property of the FIR filter is detrimental to the real-time requirements. However it can be alleviated if the sampling period *T* is small. The difficulty is that increasing the sampling frequency will result in more noisy derivatives. Therefore trade-off and compromise must be made to introduce this type of FIR in real-time applications.

5.2. Type IV FIR Differentiators

Since a FIR filter of type III has the limitation that the amplitude response must go to zero at the Nyquist frequency, it is impossible to get a full band differentiator using a finite number of coefficients. This can be shown in Fig. 3 where transition frequency range of 0.85~1.0 is associated with the 150th order (length of 301) central difference Taylor series differentiator.

A FIR filter of type IV has an even length and anti-symmetric impulse response. The type IV FIR is preferable to a type III as a differentiator in terms of the frequency response. This can be evidenced by the simplest FIR differentiator of $y_n = x_n - x_{n-1}$, which has a frequency response of

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$$

$$\Rightarrow H(\omega) = 1 - e^{-j\omega} = j \cdot 2\sin\frac{\omega}{2} \cdot e^{-j\frac{\omega}{2}} \qquad (17)$$

The corresponding amplitude response against low order Taylor series approximations is shown in Fig. 5
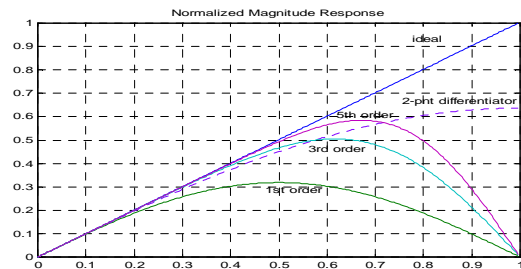

Fig. 5: Frequency response of the simplest IV differentiator against low order Taylor Series FIR filters

It can be seen that even though the differentiator is the simplest form, it is closest to the ideal at low frequencies (<0.2). It has a better amplitude response for the rest of frequency band than its type III counterpart of first-order. It also has a linear frequency response and therefore has a constant group delay at half the sampling period. The

type IV FIR differentiators are superior to the type III FIR differentiators in terms of the frequency response, since they have no disadvantageous characteristic of being zero at $\omega=1$.

Details of type IV differentiator design are referred to Chen (2001,p332). An example differentiator of length 8 (7th order) is given with the transfer function of

$$H(z) = -0.0260 + 0.0509z^{-1} - 0.1415z^{-2} + 1.2732z^{-3}$$
$$- 1.2732z^{-4} + 0.1415z^{-5} - 0.0509z^{-6} + 0.0260z^{-7}$$

$$(18)$$

which minimizes

$$E = \int_{-\pi}^{\pi} \left| H(e^{j\omega}) - j\omega e^{jN\omega/2} \right|^2 d\omega \qquad (19)$$

Therefore it is an optimal differentiator in the sense of least squares with an excellent frequency response at high frequency band. The noise amplification can be calculated from Eq. (10) as $\sigma^2 y_n = 3.2887$, which is acceptable so far.

It may be expected that a type IV FIR obtained from the Remez exchange algorithm (Parks and McClellan 1972) would be able to deliver a better performance. This is because the Remez exchange algorithm is a minimax optimal, i.e. minimize {maximum [$H_{ideal}(\omega)$-$H_{disigned}(\omega)$]} for all frequencies, and is more difficult to mathematically compute, but guarantees that the worst case error has been reduced to a quantifiable value. To verify this, the frequency responses have been depicted in Fig. 6 for the 7th and 25th-order filters respectively by the Remez algorithm
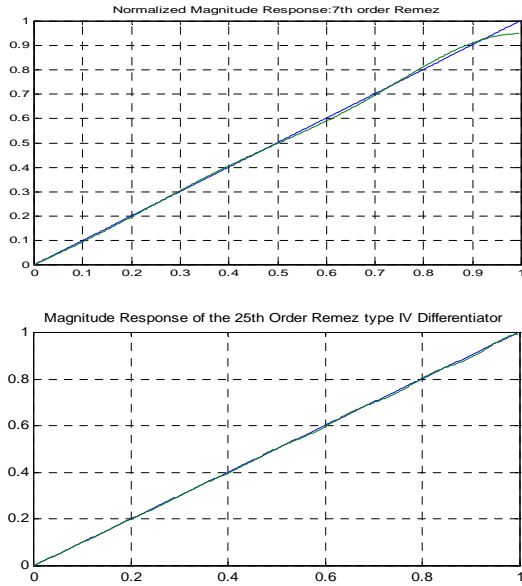

Fig. 6: Frequency response of type IV FIR filters by the Remez exchange algorithm

The FIR filter design by the Remez algorithm is referred to as the equal ripple design. This is because the method can suppress the ripples from the Gibbs phenomenon (Antoniou 1993) to a certain level and turn them into equal ripples in both the passband and stopband.

It seems that type IV FIR differentiators using the Remez exchange algorithm will give us a closing solution. However, the resultant filters provide the first derivative with system biases and higher level of noise.

Type IV FIR differentiators based on Taylor series (Khan, Ohba et al. 2000) have also been tested in this research. It has been found that wideband type IV differentiators are associated with heavy noise amplifications and big biases. Our investigation of type IV FIR filters for differentiator design is still at an early stage and continuing.

5.3. Other FIR Differentiators

In a series of publications, Kumar and Dutta (1988; 1988; 1989; 1989) presented optimal and maximally linear FIR differentiators for low-frequency, mid-frequency, and around specific frequency respectively. They gave the explicit formulae and efficient recursive algorithms to calculate the impulse response of filters. Their contributions are highly appreciated, for example, as the state of art differentiators by Al-Alaoui (1993). In the case of signals that have low frequency components contaminated by wideband noise, FIR differentiators of optimum white-noise attenuation are desired. Kavanagh (2001) investigated the impact of quantization noise on signal from systems with low-frequency rates of change. It is showed that the differentiator proposed by Vainio et al. (1997)

$$h_n = \frac{6(N-1-2n)}{N(N^2-1)} \qquad 0 \le n \le N-1 \qquad (20)$$

has an optimum white-noise attenuation and a constant group delay. Kavanagh also proposed a better differentiator for the rate experiencing slow changes

$$h_n = \begin{cases} \dfrac{1}{N-1} & n = 0 \\ 0 & 0 \le n \le N-2 \\ -\dfrac{1}{N-1} & n = N-1 \end{cases} \qquad (21)$$

This differentiator has the characteristic of minimising the worst-case error. Clearly when N=2 (type IV), this becomes the simplest two-point differentiator and when N=3 (type III), this turns into the three-point first-order differentiator of a Taylor series approximation.

6. IIR filters

There is another category of filters known as the Infinite Impulse Filter (IIR). A causal IIR filter is represented by

$$y_n = \sum_{k=0}^{N} c_k x_{n-k} + \sum_{m=1}^{M} d_m y_{n-m} \qquad (22)$$

where the output signal at a given instant is obtained as the weighted sum of the signal x_{n-k}, and the past outputs of y_{n-m}. As suggested by its name, an impulse input has a response that lasts forever since the output will be recursively used. It is the recursive characteristic that allows IIR filters to be implemented with a lesser order and better performance when compared with FIR filters. Thus IIR filters are attractive for real-time applications.

An IIR filter is unstable if its response to a transient input increases without bound. Poles and zeros are used to analyse the stability of an IIR filter. The poles are the roots of the denominator and the zeros are the roots of the numerator in the transfer function. An IIR filter is stable if and only if, all poles of H(z) are inside the unit circle on the z-plane (Stearns, 2003, p83).

The IIR filter cannot be designed by calculating the impulse response from the known frequency response as is the case in FIR designs. Many IIR filters can be derived from the analogue filter designs and then transformed into the sampled z-plane. Another popular method is the bilinear transform. The IIR differentiator design has been of considerable interest (Rabiner and Steiglitz, 1970). Among various recursive differentiator designs, Al-Alaoui's second order IIR family (1992; 1993; 1994) has been highly acknowledged and widely used, for example (Chen and Lee, 1995).The novel approach of designing digital differentiators by Al-Alaoui is an extension of the method in designing analogue differentiators by using integrators. That is, in the analogue signal processing, differentiators are often obtained by inverting the transfer functions of analogue integrators.

The general procedures to derive the Al-Alaoui family are as follows

- design an integrator that has the same range and accuracy as the desired differentiator;
- invert the obtained transfer function of the integrator;
- reflect the poles that lie outside the unit circle to inside, in order to stabilise the resultant transfer function; and
- compensate the magnitude using the reciprocals of the poles that lie outside the circle.

6.1. Al-Alaoui's First-Order Differentiator

A first-order IIR differentiator was developed by Al-Alaoui (1993) with an effective range 0.78 of the Nyquist frequency based on a non-minimum phase digital integrator. The integrator is a synthesis of the rectangular integrator and the trapezoidal integrator. By assigning weighting factors of ¾ and ¼ to the transfer functions of the integrators respectively, the ideal integrator, which has the following transfer function, is approximated

$$H_I(z) = \frac{3}{4} H_R(z) + \frac{1}{4} H_T(z)$$
$$= \frac{3}{4} \cdot \frac{T}{z-1} + \frac{1}{4} \frac{T(z+1)}{2(z-1)} = \frac{T}{8} \cdot \frac{z+7}{z-1} \qquad (23)$$

Reflecting the zero $z=-7$ with its reciprocal $-1/7$, and compensating the magnitude by multiplying $r=7$, results in a minimum phase digital integrator with the transfer function

$$H_I(z) = \frac{7 \cdot T}{8} \cdot \frac{z + \frac{1}{7}}{z-1} \qquad (24)$$

Inverting the above transfer function yields the Al-Alaoui's stabilized IIR differentiator of the first order

$$H_D(z) = \frac{\frac{8}{7}}{T} \cdot \frac{z-1}{z + \frac{1}{7}} \qquad (25)$$

The characteristics of this differentiator is shown in Fig. 7. This differentiator is able to approximate the ideal differentiator up to 0.78 of the full band, and has an outstanding "linear phase" response. Al-Alaoui reported that within the effective frequency range, it has a less than 2.0% magnitude error. Since it is of first-order, the delay of the filter is just half of the sample thus it meets every requirement to be used in real-time.
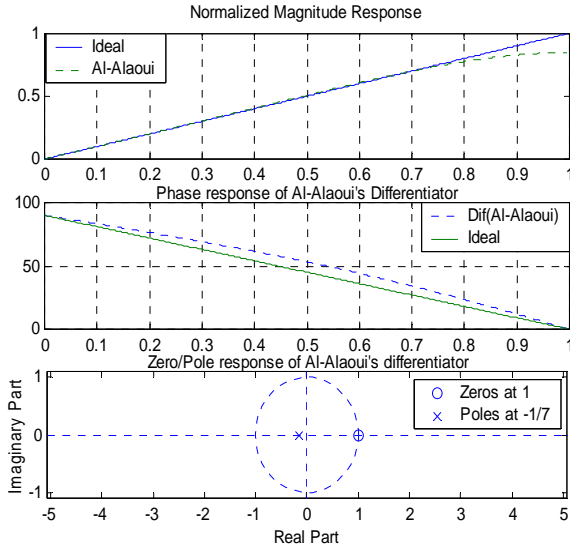
Fig. 7: Characteristics of the first order IIR differentiator

6.2. First-order IIR Differentiator Family

Al-Alaoui contributes the above differentiator as an individual. However, a family of such first order differentiators can be derived following his methodology. That is, while Al-Alaoui designates the weighting factors of ¾ and ¼ empirically, we may get the optimal weights experimentally. To achieve this, a variable α is introduced to adjust the weighting factor in the way of

$$H_I(z) = \alpha H_R(z) + (1-\alpha)H_T(z)$$

$$= \alpha \cdot \frac{T}{z-1} + (1-\alpha)\frac{T(z+1)}{2(z-1)} = \frac{T(1-\alpha)\left[z + \frac{1+\alpha}{1-\alpha}\right]}{2(z-1)}$$

(26)

where $0<\alpha<1$ serves as a tuner to adjust the integrator so that it better closes to the ideal. $\alpha = \text{¾}$ can be used as a good reference to refine the integrator in the desired range of frequencies. Obviously it has a zero outside the unit circle. Applying Al-Alaoui's procedure to reflect the zero with its reciprocal and to compensate the magnitude, a variable integrator is obtained as

$$H_I(z) = \frac{T(1+\alpha)\left[z + \frac{1-\alpha}{1+\alpha}\right]}{2(z-1)}$$

(27)

Inverting the transfer function gives a new set of differentiators with transfer functions as

$$H_D(z) = \frac{2(z-1)}{T(1+\alpha)\left[z + \frac{1-\alpha}{1+\alpha}\right]}$$

(28)

Since $1-\alpha < 1+\alpha$, the pole is well located inside the unit circle and the resultant differentiators are, therefore, stable. Setting $\alpha = \text{¾}$ gives the transfer function proposed by Al-Alaoui, and slightly changing α around ¾ results in differentiators which outperform in target bandwidth. The noise amplification of this kind of differentiator can be evaluated using Eq. (10), which is only slightly noisier than the simplest two-point differentiator.

7. Conclusions

The general theory on digital filter design has been introduced. The aim of this research is to find appropriate differentiators that can be used to derive Doppler shifts/Doppler rates from GPS observables in real-time, dynamic applications.

It is concluded that the differentiators obtained from both curve fitting and Kalman filtering require intensive computation and are lowpass. Thus they are not suitable for real-time dynamic applications. Type III FIR differentiators have the inherent nature in frequency response of approaching zero at Nyquist frequency. To extend the performance of type III FIR filters in the higher frequency bands, one has to increase the filter taps. This causes difficulties in managing the data since there are cycle-slips and loss-of-lock signals. It also results in a longer group delay that is detrimental for real-time applications where instant response is desired.

Type IV FIR differentiators using Fourier series have been found to have outstanding frequency response, however, they are noisy and biased. It is found that only the Kavanagh's differentiators of type IV deliver good first derivatives. However, they approximate the ideal differentiator only at low frequencies (lower than 0.2 of Nyquist frequency). Type III FIR filters can be used to derive Doppler/Doppler rate "observables" in post processing mode. Higher order central difference approximations using Taylor series might outperform windowed Fourier series since there is no truncation and the associated Gibbs phenomenon.

It is demonstrated that IIR filters are more favourable for real-time application. Since the outputs of the filter are recursively used, they have much lower orders than the FIR filters. The first-order IIR differentiator from Al-Alaoui is ideal in terms of the frequency response, phase linearity and half sample group delay. The proposed class of first-order IIR differentiators allows us to choose an optimum in the desired frequency range.

It is suggested that Kavanagh's differentiators can be used in static or in constant velocity modes. The proposed first-order IIR differentiators can be adaptively used when systems experience high dynamics.

ACKNOWLEDGEMENTS

REFERENCES

Al-Alaoui M.A. (1992): *Novel Approach to Designing Digital Differentiators*. Electronics Letters 28(15): 1376-1378.

Al-Alaoui M.A. (1993): *Novel Digital Integrator and Differentiator*. Electronics Letters 29(4): 1376-1378.

Al-Alaoui M.A. (1994): *Novel IIR Differentiator from the Simpson Integration Rule*. IEEE Transactions on Circuits and System-I Fundamental , Theory and Applications 41(2).

Antoniou A. (1993): *Digital Filters: Analysis, Design and Appications*. New York, McGraw-Hill 0-07-002121X.

Bona P. (2000): *Precision, Cross Correlation, and Time Correlation of GPS Phase and Code Observations*. GPS solutions 4(2): 3-13.

Borre K. and Tiberius C. (2000): *Time Series Analysis of GPS Observables*. proceedings of the ION GPS 2000, Salt Lake City, UT, 1885-1894.

Brown R.G. and Hwang P.Y.C. (1992): *Introduction to Random Signals and Applied Kalman Filtering*. New York, John Wiley &Sons, INC 0471-52573-1.

Bruton A.M., Glennie C.L. and Schwarz K.P. (1999): *Differentiation for High-Precision GPS Velocity and Acceleration Determination*. GPS solutions 2(4): 7-21.

Cannon M.E., Lachapelle G., Szarmes M.C., Hebert J.M., Keith J. and Jokerst S. (1998): *DGPS Kinematic Carrier Phase Signal Simulation Analysis for Precise Velocity and Position Determination*. Journal of the Institute of Navigation 44(2): 231-245.

Carlsson B., Ahlen A. and Sternad M. (1991): *Optimal Differentiation Based on Stochastic Signal Models*. IEEE Transactions on Signal Processing 39(2): 341-353.

Chen C.-K. and Lee J.-H. (1995): *Design of High-Order Digital Differentiators using L1 Error Criteria*. IEEE Transactions on Circuits and Systems II: Analogy and Digital Signal Processing 42(4): 287-291.

Chen C.-T. (2001): *Digital Signal Processing: Spectral Computation and Filter Design*. New York, Oxford, Oxford University Press 0-19-513638-1.

Farlex (2004): *TheFreeDictionaryCom*: Kaiser Window. http://encyclopedia.thefreedictionary.com/Kaiser%20window.

Fenton P. and Townsend B. (1994): *NowAtel Communications Ltd,-What's New?* Proceedings of KIS'94, Banff, AB, CA, 25-29.

Hamming R.W. (1977): *Digital Filters*. Englewood Cliffs, Prentice-Hall, Inc. 0-13-212571-4.

Harvey N. (2004): *Doppler and Carrier Phase Measures in Trimble 5700 GPS Receiver*. Christchurch, New Zealand (Personal communication).

Hebert C.J. (1997): *High Accuracy GPS Velocity Using the Carrier Phase Observable*. Proceedings of KIS'97, Banff, AB, Canada, 265-269.

Hebert C. J., Keith J., Ryan S., Szarmes M., Lachapelle G. and Canon M. E. (1997): *DGPS Kinematic Carrier Phase Signal Simulation Analysis for Precise Aircraft Velocity Determination*. Proceedings of the ION GPS-97, Albuquerque, NM,

Jekeli C. and Garcia R.. (1997): *GPS Phase Accelerations for Moving-base Vector Gravimetry*. Journal of Geodesy 71(10): 630-639.

Kavanagh R.C. (2001): *FIR Differentiators for Quantized Signals*. IEEE Transactions on Signal Processing 49(11): 2713-2720.

Kennedy S. (2002): *Precise Acceleration Determination from Carrier Phase Measurements*. Proceeding of the ION GPS-2002, Portalnd Oregon, USA, 962-972.

Kennedy S.L. (2003): *Precise Acceleration Determination from Carrier-Phase Measurements*. Navigation, Journal of the Institute of Navigation 50(1): 9-19.

Khan I.R. and Ohba R. (1999): *New Design of Full Band Differentiators Based on Taylor Series*. IEEE Proceedings-Vision, Image and Signal Processing 146(4): 185-189.

Khan I. R., Ohba R. and Hozumi N. (2000): *Mathematical Proof of Explicit Formulas for Tap-coefficients of Full-band FIR Digital Differentiators*. IEE Proceedings-Vision, Image and Signal Processing 147(6): 553-555.

Kumar B. and Dutta S. C. R. (1988): *Coefficients of Maximally Linear, FIR Digital Differentiators for Low Frequencies*. Electronics Letters 24(9): 563-565.

Kumar B. and Dutta S.C.R. (1988): *Design of Digital Differentiators for Low Frequencies*. Proceedings of the IEEE 76(3): 287-289.

Kumar B. and Dutta S.C.R. (1989): *Design of Efficient FIR Digital Differentiators and Hilbert Transforms for Midband Frequency Ranges*. Internal 36(6).

Kumar B. and Dutta S.C.R. (1989): *Maximally Linear FIR Digital Differentiators for Midband Frequencies*. Int. J. Circuit Theory Appl. 17: 483-488.

Parks T.W. and McClellan J.J. (1972): ***Chebyshev Approximatin for Nonrecursive Digital Filters with Linear Phase.*** IEEE Trans. Circuit Theory 19: 189-194.

Rabiner L. and Steiglitz K. (1970): ***The Design of Wide-band Recursive and Nonrecursive Digital Differentiators***. IEEE Transactions on Audio and Electroacoustics 18(2): 204-209.

Selesnick I.W. (2002): ***Maximally Flat Low-pass Digital Differentiator***. IEEE Transactions on Circuits and Systems II: Analogue and Digital Signal Processing 49(3): 219.

Stearns S.D. (2003): ***Digital Signal Processing with Examples in MATLAB.*** Boca Raton, London, New York, Washington,D.C., CRC press 0-8493-1091-1.

SuperstarII (2004): ***Superstar II Product Brochure***. 2004 http://www.seabed.nl/pdf/superstar.pdf.

Vainio O., Renfors M. and SaramT. (1997): ***Recursive Implementation of FIR Differentiators with Optimum Noise Attenuation***. IEEE Transactions on Instrum. Meas. 46: 1202-1207.

Wolfe J.D., Williamson W.R. and Speyer J.L. (2003): ***Hypothesis Testing for Resolving Integer Ambiguity in GPS. Navigation***, Journal of the Institute of Navigation 50(1): 45-56.