

# Differential GPS: the reduced-difference approach

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**Abstract.** In the traditional approach to differential GNSS, the satellite error terms are eliminated by forming the so-called single differences (SD). One then gets rid of the receiver error terms by computing, for each receiver to be considered, the corresponding double differences (DD): the discrepancies between the single differences (SD) and one of them taken as reference. To handle the SD's in a homogeneous manner, one may equally well consider the discrepancies between the SD's and their mean value. In this paper, these 'centralized differential data' are referred to as 'reduced differences' (RD). In the case where the GNSS device includes only two receivers, this approach is completely equivalent to 'double centralization.' More precisely, the information contained in the 'double centralized observations' is then a simple anti-symmetric transcription of that contained in the reduced differences. The ambiguities are then rational numbers which are related to the traditional integer ambiguities in a very simple manner. The properties established in this paper shed a new light on the corresponding analysis. (The extension to GNSS networks with missing data will be presented in a forthcoming paper.) The corresponding applications concern the identification of outliers in real time. Cycle slips combined with miscellaneous SD biases can thus be easily identified.

**Key words.** GNSS, DGPS, centralized undifferential methods, RTK. Data assimilation, DIA.

## 1 Introduction

The global positioning techniques are based on the following observational equations. For each frequency  $f_\nu$ , for each receiver-satellite pair  $(r, s)$ , and at each epoch  $t$ , the code and carrier-phase data are respectively of the form (e.g., Sect. 14 in Strang and Borre 1997)

$$p_{\nu,t}(r, s) = \rho_t(r, s) + c[dt_{\nu,t}(r) - dt_{\nu,t}(s)] + \epsilon_{\nu,t}(r, s) \quad (1)$$

$$\phi_{\nu,t}(r, s) = \rho_t(r, s) + c[\delta t_{\nu,t}(r) - \delta t_{\nu,t}(s)] + \lambda_\nu[\varphi_{\nu,0}(r) - \varphi_{\nu,0}(s)] + \lambda_\nu N_\nu(r, s) + \epsilon_{\nu,t}(r, s) \quad (2)$$

In these equations, which are expressed in length units,  $\rho_t(r, s)$  is the receiver-satellite range: the distance between satellite  $s$  (at the time  $t - \tau$  where the signal is emitted) and receiver  $r$  (at the time  $t$  of its reception). Clearly, the  $\lambda_\nu$ 's denote the wavelengths of the carrier waves; the rational integers  $N_\nu(r, s)$  are the integer carrier-phase ambiguities. The instrumental delays and clock errors that for a given  $(\nu, t)$  depend only on  $r$  and  $s$  are lumped together in the receiver and satellite error terms  $dt_{\nu,t}(r)$ ,  $dt_{\nu,t}(s)$  for the code, and  $\delta t_{\nu,t}(r)$ ,  $\delta t_{\nu,t}(s)$  for the phase ( $c$  is the speed of light);  $\varphi_{\nu,0}(r)$  and  $\varphi_{\nu,0}(s)$  are the initial phases (expressed in cycles) in receiver  $r$  and satellite  $s$ , respectively. Here, for clarity, the ionospheric and tropospheric delays are ignored. At this introductory level, we thus consider that the data have been corrected for these delays. Clearly, the code and phase errors  $\epsilon_{\nu,t}(i, j)$  and  $\epsilon_{\nu,t}(i, j)$  include both noise and residual model errors.

For our present purposes, we now concentrate on Equation (2) in the single-frequency mode:

$$\phi_t(r, s) = \rho_t(r, s) + c[\delta t_t(r) - \delta t_t(s)] + \lambda[\varphi_0(r) - \varphi_0(s)] + \lambda N(r, s) + \epsilon_t(r, s) \quad (3)$$

In what follows, a notation such as  $a := b$  means 'a is equal to b by definition.' Let  $r_1$  now be the reference receiver, and  $r_2$  be that of the user. Denote by  $s_1, s_2, \dots, s_n$  the satellites involved in the GPS device. A quantity such as

$$\vartheta^j := \vartheta(r_2, s_j) - \vartheta(r_1, s_j) \quad (4)$$

is then referred to as a single difference (SD) in  $\vartheta$ . By using this notation, Equation (3) then yields

$$\phi_t^j = \rho_t^j + c[\delta t_t(r_2) - \delta t_t(r_1)] + \lambda[\varphi_0(r_2) - \varphi_0(r_1)] + \lambda a^j + \epsilon_t^j \quad (5)$$

where

$$a^j := N^j \quad (6)$$

One thus gets rid of the satellite error terms. The  $a^j$ 's are the integer ambiguities of the SD phase data.

## 1.1 Basic notions

### 1.1.1 Double differences

In the traditional approach to differential GNSS, one first selects a reference satellite. Here, this satellite is denoted by  $s_k$ . A quantity such as

$$\vartheta_k^j := \vartheta^j - \vartheta^k \quad (i \neq j) \quad (7)$$

is then referred to as a double difference (DD) in  $\vartheta$  (see Fig. 1).

By subtracting from Eq. (5) its expression for  $j = k$  (term by term), one then obtains the relation

$$\phi_{t;k}^j = \rho_{t;k}^j + \lambda a_k^j + \varepsilon_{t;k}^j \quad (a_k^j \in \mathbb{Z}) \quad (8)$$

One thus gets rid of the receiver error terms. The  $a_k^j$ 's are said to be the DD integer ambiguities of the problem.

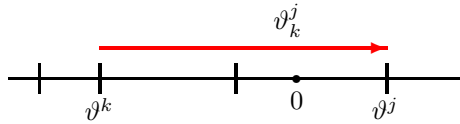


Fig. 1 *Notion of double difference.* The double difference  $\vartheta_k^j$  is the value of the single difference  $\vartheta^j$  by taking as origin (or reference) the value of the single difference  $\vartheta^k$ .

### 1.1.2 Reduced differences

In the approach presented in this paper, we consider a homogeneous way of eliminating the receiver error terms. The idea is to consider the quantities (see Fig. 2)

$$\vartheta_0^j := \vartheta^j - \vartheta^0 \quad (9)$$

where  $\vartheta^0$  is the mean value of the  $\vartheta^j$ :

$$\vartheta^0 := \frac{1}{n} \sum_{j=1}^n \vartheta^j \quad (10)$$

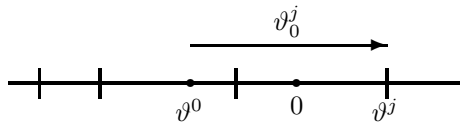


Fig. 2 *Notion of reduced difference.* The reduced difference  $\vartheta_0^j$  is the value of the single difference  $\vartheta^j$  by taking as origin (or reference) the mean value  $\vartheta^0$  of the single differences (compare with Fig. 1).

Clearly, this barycentric value can be regarded as a virtual SD associated with a virtual reference satellite  $s_0$ . According to a well-known barycentric property (for further details see Sect. 3), for any  $k$ , we have

$$\sum_{j=1}^n |\vartheta_0^j|^2 \leq \sum_{j \neq k} |\vartheta_k^j|^2 \quad (11)$$

The  $\vartheta_0^j$ 's can therefore be referred to as 'reduced differences' (RD).

Subtracting from Eq. (5) its expression in terms of mean values (term by term), we then obtain the relation similar to Eq. (8)

$$\phi_{t;0}^j = \rho_{t;0}^j + \lambda a_0^j + \varepsilon_{t;0}^j \quad (a_0^j \in \mathbb{Q}) \quad (12)$$

Note that the RD ambiguities  $a_0^j$ 's are rational numbers (and not in general rational integers).

### 1.1.3 Differential observations

By construction, the DD's of the function

$$\vartheta_d(r_i, s_j) := \begin{cases} 0 & \text{if } i = 1 \text{ or } j = k; \\ \vartheta_k^j & \text{otherwise.} \end{cases} \quad (13)$$

are the DD's of the function  $\vartheta(r_i, s_j)$ . Such a function can therefore be referred to as a 'differential observational' (DO) function.

### 1.1.4 Reduced observations

By construction, the SD's of the function

$$\vartheta_r(r_i, s_j) := \begin{cases} 0 & \text{if } i = 1; \\ \vartheta_0^j & \text{otherwise.} \end{cases} \quad (14)$$

are the reduced differences of the function  $\vartheta(r_i, s_j)$ . Such a function can therefore be referred to as a 'reduced observational' (RO) function.

### 1.1.5 Centralized observations

In the 'centralized observational approach' of Shi and Han (1992), one gets rid of the satellite error terms by forming the single centralized observations

$$\begin{aligned} \vartheta_c^{(1)}(r_i, s_j) &= \vartheta(r_i, s_j) - \frac{1}{2} \sum_{i=1}^2 \vartheta(r_i, s_j) \\ &= (-1)^i \frac{1}{2} [\vartheta(r_2, s_j) - \vartheta(r_1, s_j)] \\ &= (-1)^i \frac{\vartheta^j}{2} \end{aligned}$$

The receiver error terms are then eliminated by forming the double centralized observations

$$\begin{aligned} \vartheta_c^{(2)}(r_i, s_j) &= \vartheta_c^{(1)}(r_i, s_j) - \frac{1}{n} \sum_{j=1}^n \vartheta_c^{(1)}(r_i, s_j) \\ &= (-1)^i \frac{1}{2} \left( \vartheta^j - \frac{1}{n} \sum_{j=1}^n \vartheta^j \right) \\ &= (-1)^i \frac{\vartheta_0^j}{2} \end{aligned}$$

One is then led to say that

$$\vartheta_c(r_i, s_j) := (-1)^i \frac{\vartheta_0^j}{2} \quad (15)$$

is a centralized observational function. In the case where the GNSS device includes only two receivers, the information contained in the centralized observations is therefore a simple ‘antisymmetric transcription’ of that contained in the reduced differences. The title of the paper was chosen accordingly.

## 1.2 Contents

The theoretical framework of this contribution is presented in Sect. 2. As clarified in Sect. 3, the DD and RD approaches prove to be equivalent. In particular, although the RD ambiguities are rational numbers, the ambiguity problems to be solved are the same. Section 5.2 is devoted to this point. The RD approach should however be preferred. Indeed, as shown in Sect. 4, it reveals interesting properties which give a deeper insight into the problem. These properties, which are masked in the DD approach, shed a new light on the centralized undifferential method of Shi and Han (1992). They also complete the dual algebraic approach of Lannes and Durand (2003). As a result, these equivalent approaches can benefit from each other. As shown in Sect. 6, one of these properties plays a key role in the DIA procedures of the data assimilation processes presented in Sect. 5. The SD biases, among which the cycle slips (if any), can then be identified in real time. Related comments are to be found in Sect. 7. As pointed out in that section, the analysis presented in this paper can be regarded as an introduction to the case of GNSS networks with missing data (Lannes 2008).

## 2 Theoretical framework

In the context defined in Sect. 1, the notion of observational space can be specified as follows.

### 2.1 Observational spaces

In what follows, it may be convenient to consider that a function such as  $\vartheta(r, s)$  takes its values on a rectangular grid. When the GNSS device includes two receivers and  $n$  satellites, this grid includes two lines and  $n$  columns;  $\vartheta$  is then regarded as a vector of the observational space  $E := \mathbb{R}^{2n}$ . Clearly, these values are the components of  $\vartheta$  in the standard basis of  $E$ . The notation  $E_\psi$  specifies the nature of the vectors  $\vartheta$  of  $E$ :  $\psi = p$  for the code,  $\psi = \phi$  for the phase. The variance-covariance matrix of the corresponding data vector is denoted by  $\mathbf{V}_\psi$ ;  $V_\psi$  is the operator on  $E$  induced by  $\mathbf{V}_\psi$ . One is then led to define the ‘observational data space’ of type  $\psi$  as the space  $E_\psi^+$  with inner product

$$\langle \vartheta | \vartheta' \rangle_{E_\psi^+} := (\vartheta \cdot V_\psi^{-1} \vartheta')_E \quad (16)$$

Clearly,  $E^+ \equiv E_\psi^+$  is a real Hilbert space.

#### 2.1.1 Nuisance delay space

In what follows, the space  $E_0$  of functions  $\vartheta(r_i, s_j)$  of the form  $\varphi(s_j) - \varphi(r_i)$  is referred to as the nuisance delay space (see Eqs. (1), (2) and Fig. 4). In the special case under consideration (with two receivers), this subspace of  $E$  is of dimension  $n + 1$ .

#### 2.1.2 Clean observational spaces

The orthogonal complement of  $E_0$  in  $E$ , denoted by  $E_c$ , is referred to as the ‘clean observational (CO) space.’ The orthogonal complement of  $E_0$  in  $E^+$ ,  $E_c^+$ , is then referred to as the ‘CO data space’ (see Fig. 4). In the special case under consideration,  $E_c$  and  $E_c^+$  are of dimension  $2n - (n + 1) = n - 1$ . As shown below,  $E_c$  is then the space of ‘centralized observational functions’  $\vartheta_c$  defined by Eq. (15).

**Proof.** In the Euclidean space  $E$ ,  $\vartheta_c$  is orthogonal to any nuisance function of  $E_0$ . Indeed,

$$\begin{aligned} \sum_{i=1}^2 \sum_{j=1}^n [\varphi(s_j) - \varphi(r_i)] (-1)^i \vartheta_0^j &= \sum_{j=1}^n \vartheta_0^j \varphi(s_j) \sum_{i=1}^2 (-1)^i \\ &\quad - \sum_{i=1}^2 (-1)^i \varphi(r_i) \sum_{j=1}^n \vartheta_0^j \end{aligned}$$

with  $\sum_{i=1}^2 (-1)^i = 0$  and  $\sum_{j=1}^n \vartheta_0^j = 0$ . The property then follows from the fact that the functions  $\vartheta_c$  form a space of dimension  $n - 1$ .  $\square$

**Remark 2.1.2.** Let  $P_c$  be the orthogonal projection of  $E$  onto  $E_c$ . Here, the centralized observational function  $\vartheta_c$  defined by Eq. (15) is the projection of  $\vartheta$  on  $E_c$ :  $\vartheta_c := P_c \vartheta$ . In other terms, the ‘cleaning operator’  $P_c$  then reduces to the ‘double-centralization operator.’ This does not hold for GNSS networks with missing data. The terminology was chosen accordingly. Depending on the context, C and subscript c stand for ‘clean’ or ‘centralized.’

### 2.2 SD space

Denoting by  $\mathbf{b} := \{\mathbf{e}_j\}_{j=1}^n$  the standard basis of  $\mathbb{R}^n$ , let us consider the vector  $\boldsymbol{\vartheta} := \sum_{j=1}^n \vartheta^j \mathbf{e}_j$  in which the  $\vartheta^j$ ’s are the single differences defined in Eq. (4). Clearly, such a vector can be regarded as a SD vector. In this context, we say that  $F := \mathbb{R}^n$  is the ‘SD space’ (see Fig. 3).

#### 2.2.1 SD operator

The SD operator is the operator from  $E^+$  into  $F$  defined by the relation (see Eq. (4))

$$S\vartheta := \boldsymbol{\vartheta} \quad \text{i.e.} \quad (S\vartheta)^j := \vartheta^j \quad (17)$$

We now denote by  $S^\dagger$  the corresponding ‘backprojection’ operator, i.e., the operator from  $F$  into  $E$

$$(S^\dagger \boldsymbol{\vartheta})(r_i, s_j) := (-1)^i \vartheta^j \quad (18)$$

For any  $\boldsymbol{\vartheta} \in F$ , the function  $\vartheta' := S^\dagger \boldsymbol{\vartheta} / 2$  is such that  $S\vartheta' = \boldsymbol{\vartheta}$ ;  $S$  is therefore surjective.

In what follows,  $\mathbf{V}_\psi$  is the variance-covariance matrix of the SD data vector  $\boldsymbol{\psi} := S\boldsymbol{\vartheta}$ . Denoting by  $V_\psi$  the operator on  $F$  induced by  $\mathbf{V}_\psi$ , we have

$$V_\psi = S V_\psi S^\dagger \quad (19)$$

We now show that the adjoint of  $S$  is given by the relation

$$S^* = V_\psi S^\dagger \quad (20)$$

hence, from Eq. (19),

$$S S^* = V_\psi \quad (21)$$

**Proof.** By definition,  $S^*$  is the operator from  $F$  into  $E^+$  such that for any  $\vartheta' \in E^+$  and any  $\boldsymbol{\vartheta} \in F$ , we have  $\langle S\vartheta' \cdot \boldsymbol{\vartheta} \rangle_F = \langle \vartheta' | S^* \boldsymbol{\vartheta} \rangle_{E^+}$ . Clearly,

$$\langle S\vartheta' \cdot \boldsymbol{\vartheta} \rangle_F = \sum_{j=1}^n \left[ \sum_{i=1}^2 (-1)^i \vartheta'(r_i, s_j) \right] \vartheta^j$$

From Eq. (18), we therefore have

$$\langle S\vartheta' \cdot \boldsymbol{\vartheta} \rangle_F = \sum_{i=1}^2 \sum_{j=1}^n \vartheta'(r_i, s_j) (S^\dagger \boldsymbol{\vartheta})(r_i, s_j)$$

i.e.,  $\langle S\vartheta' \cdot \boldsymbol{\vartheta} \rangle_F = \langle \vartheta' \cdot S^\dagger \boldsymbol{\vartheta} \rangle_E = \langle \vartheta' \cdot [V_\psi^{-1} V_\psi] S^\dagger \boldsymbol{\vartheta} \rangle_E$ . As a result,

$$\langle S\vartheta' \cdot \boldsymbol{\vartheta} \rangle_F = \langle \vartheta' | V_\psi S^\dagger \boldsymbol{\vartheta} \rangle_{E^+}$$

hence  $S^* = V_\psi S^\dagger$ .  $\square$

### 2.2.2 RD space and RD ambiguity lattice

Let us denote by  $F_0$  the space of vectors  $\boldsymbol{\vartheta} \in F$  whose components  $\vartheta^j$  are identical. The orthogonal complement of  $F_0$  into  $F$  is the space (see Fig. 3)

$$F_r := \{ \boldsymbol{\vartheta} \in F : \sum_{j=1}^n \vartheta^j = 0 \}$$

As  $F_0$  is a one-dimensional space,  $F_r$  is of dimension  $n-1$ .

Let  $Q_0$  and  $Q_r$  be the orthogonal projections of  $F$  onto  $F_0$  and  $F_r$ , respectively. Clearly, these operators are explicitly defined by the relations

$$(Q_0 \boldsymbol{\vartheta})^j = \vartheta^0 \quad (Q_r \boldsymbol{\vartheta})^j = \vartheta^j - \vartheta^0 \quad (22)$$

where  $\vartheta^0$  is the mean value of the  $\vartheta^j$ 's. With regard to the RD approach, we are then led to set (see Eqs. (9), (10), (22) and Fig. 3)

$$\boldsymbol{\vartheta}_r := Q_r \boldsymbol{\vartheta} \quad (23)$$

Clearly, the components of  $\boldsymbol{\vartheta}_r$  in basis  $\mathbf{b}$  are the  $n$  reduced differences  $\vartheta_0^j$ ;  $F_r$  can therefore be referred to as the 'RD space.' Note that  $\vartheta_c$  is related to  $\boldsymbol{\vartheta}_r$  by the relation  $\vartheta_c = S^\dagger \boldsymbol{\vartheta}_r / 2$  (see Eqs. (15) and (18)).

As  $\boldsymbol{\vartheta}_r$  is the projection of  $\boldsymbol{\vartheta}$  on  $E_r$ , we have

$$\sum_{j=1}^n |\vartheta_0^j|^2 = \sum_{j=1}^n |\vartheta^j - \vartheta^0|^2 = \inf_{\vartheta^0 \in \mathbb{R}} \sum_{j=1}^n |\vartheta^j - \vartheta^0|^2$$

The property expressed in Eq. (11) results from this relation.

The projection of  $\mathbb{Z}^n$  onto  $F_r$  is a lattice of rank  $n-1$ : the 'RD ambiguity lattice'  $L_r$  (see Fig. 3). In basis  $\mathbf{b}$  (which is not a basis of  $L_r$ ), the components of a point  $\mathbf{a}_r$  of  $L_r$  are rational numbers: the  $n$  rational ambiguities  $a_0^j$ .

**Remark 2.2.2.** The RO functions  $\vartheta_r$  defined by Eq. (14) form a subspace of  $E$  denoted by  $E_r$  (see Fig. 4). Clearly, this 'RO space' is a simple insertion of  $F_r$  in  $E$ .

### 2.2.3 DD space and DD ambiguity lattice

In the DD approach,  $k$  being fixed, one is led to consider the subspace of  $F$  (see Fig. 3)

$$F_d := \{ \boldsymbol{\vartheta} \in F : \vartheta^k = 0 \}$$

By construction,  $F_d$  is isomorphic to  $\mathbb{R}^{n-1}$ . Let  $Q_d$  now be the oblique projection of  $F$  onto  $F_d$  along  $F_0$ . Note that  $Q_d$  is explicitly defined by the relation

$$(Q_d \boldsymbol{\vartheta})^j = \vartheta^j - \vartheta^k \quad (24)$$

We are then led to set (see Eq. (7) and Fig. 3)

$$\boldsymbol{\vartheta}_d := Q_d \boldsymbol{\vartheta} \quad (25)$$

Let  $\mathbf{b}_d := \{\mathbf{e}_j\}_{j \neq k}$  be the standard basis of  $F_d$ . As the components of  $\boldsymbol{\vartheta}_d$  in basis  $\mathbf{b}_d$  are the  $n-1$  double differences  $\vartheta_k^j$ ,  $F_d$  can be regarded as a 'DD space.'

The intersection of  $\mathbb{Z}^n$  with  $F_d$  is a lattice of rank  $n-1$ : the 'DD ambiguity lattice'  $L_d$  (see Fig. 3). In basis  $\mathbf{b}_d$ , the components of a point  $\mathbf{a}_d$  of  $L_d$  are rational integers: the  $n-1$  integer ambiguities  $a_k^j$  ( $j \neq k$ ).

Clearly,  $L_d = Q_d L_d$  hence  $L_d \subset Q_d \mathbb{Z}^n$  (since  $L_d$  is a subset of  $\mathbb{Z}^n$ ). Furthermore,  $Q_d \mathbb{Z}^n \subset L_d$ . We therefore have  $L_d = Q_d \mathbb{Z}^n$ . As  $L_r := Q_r \mathbb{Z}^n$  and  $Q_r = Q_r Q_d$ , it follows that  $L_r = Q_r L_d$  (see Fig. 3).

**Remark 2.2.3.** The DO functions  $\vartheta_d$  defined by Eq. (13) form a subspace of  $E$  denoted by  $E_d$  (see Fig. 4). Clearly, this 'DO space' is an insertion of  $F_d$  in  $E$ .

## 2.3 RD and DD operators

The RD operator is the operator from  $E^+$  into  $F_r$  defined by the relation

$$S_r := Q_r S \quad (26)$$

Note that  $S_r$  is surjective. (The argument is the same as that used for  $S$ .) As expected, the null space of  $S_r$  (denoted by  $\ker S_r$ ) is the nuisance delay space  $E_0$ . This property can be explicitly established as follows.



From Eqs. (33) and (35),  $\langle \boldsymbol{\vartheta}'_d | \boldsymbol{\vartheta}_d \rangle_{F_{d+}} = (\boldsymbol{\vartheta}'_d \cdot R^\dagger \boldsymbol{\vartheta}_{r+})_F$ ; hence  $\langle \boldsymbol{\vartheta}'_d | \boldsymbol{\vartheta}_d \rangle_{F_{d+}} = (R\boldsymbol{\vartheta}'_d \cdot \boldsymbol{\vartheta}_{r+})_F$ . We thus have

$$\langle \boldsymbol{\vartheta}'_d | \boldsymbol{\vartheta}_d \rangle_{F_{d+}} = \langle \boldsymbol{\vartheta}'_r | \boldsymbol{\vartheta}_r \rangle_{F_{r+}}$$

where  $\boldsymbol{\vartheta}'_r = R\boldsymbol{\vartheta}'_d$  and  $\boldsymbol{\vartheta}_r = R\boldsymbol{\vartheta}_d$ . In particular,

$$\|\boldsymbol{\vartheta}_d\|_{F_{d+}}^2 = \|\boldsymbol{\vartheta}_r\|_{F_{r+}}^2 \quad (\text{for } \boldsymbol{\vartheta}_r = R\boldsymbol{\vartheta}_d) \quad (37)$$

The DD and RD approaches are therefore ‘completely equivalent.’ This said, as shown in Sect. 4 (see, in particular, Result 4.2.2), the RD approach reveals interesting properties which are completely hidden in DD mode (see Remark 4.2.2).

## 4 Observational equivalence: Duality

### 4.1 Projection onto the CO data space

Let  $\vartheta$  be some point in the observational space  $E$ . In what follows,  $\vartheta_c^+$  denotes the orthogonal projection of  $\vartheta$  on the CO data space  $E_c^+$  (see Sect. 2.1.2 and Fig. 4):

$$\vartheta_c^+ := P_c^+ \vartheta \quad (38)$$

Clearly,  $P_c^+$  is the corresponding orthogonal projection. Let  $\boldsymbol{\vartheta}_r := S_r \vartheta$  now be the RD vector of  $\vartheta$ . The solutions of the equation  $S_r \vartheta' = \boldsymbol{\vartheta}_r$  are defined up to a vector of  $E_0$ ;  $\vartheta_c^+$  is the solution with smallest norm in  $E^+$ . The operator that maps  $\boldsymbol{\vartheta}_r$  to  $\vartheta_c^+$  is referred to as the Moore-Penrose pseudoinverse of  $S_r$ . This operator is denoted by  $S_r^+$ :

$$\vartheta_c^+ = S_r^+ \boldsymbol{\vartheta}_r \quad (39)$$

Likewise, for  $\boldsymbol{\vartheta}_d = D\boldsymbol{\vartheta}_r$ , we have  $\vartheta_c^+ = S_d^+ \boldsymbol{\vartheta}_d$ . Clearly,  $\vartheta_c^+$  can be regarded as the expression for  $\boldsymbol{\vartheta}_r$  (or  $\boldsymbol{\vartheta}_d$ ) brought back to  $E^+$  via  $S_r^+$  (or  $S_d^+$ ). In this context, we define  $\vartheta_{c+}$  as follows (see Eqs. (34), (18) and Fig. 4):

$$\vartheta_{c+} := S^\dagger \boldsymbol{\vartheta}_{r+} \quad (40)$$

The following property then completes the analysis presented in Sect. 3.

**Property 4.1.** One has  $\vartheta_c^+ = V_\psi \vartheta_{c+}$ . As a corollary,  $\|\vartheta_c^+\|_{E^+}^2 = (\vartheta_c^+ \cdot \vartheta_{c+})_E = \|\boldsymbol{\vartheta}_r\|_{F_{r+}}^2$ .

**Proof.** As  $S_r$  is surjective, its pseudoinverse is given by the relation

$$S_r^+ = S_r^* (S_r S_r^*)^{-1}$$

For any  $\boldsymbol{\vartheta}$  in  $F_r$ , we have (since  $S_r = Q_r S$ )

$$S_r^* \boldsymbol{\vartheta} = (Q_r S)^* \boldsymbol{\vartheta} = S^* Q_r^* \boldsymbol{\vartheta} = S^* Q_r \boldsymbol{\vartheta} = S^* \boldsymbol{\vartheta}$$

where  $S^* = V_\psi S^\dagger$  (Eq. (20)). As a result (see Eqs. (21) and (32)),

$$S_r S_r^* \boldsymbol{\vartheta} = Q_r S S^* \boldsymbol{\vartheta} = Q_r V_\psi \boldsymbol{\vartheta} = V_{\psi_r} \boldsymbol{\vartheta}$$

It then follows that

$$S_r^+ = S_r^* V_{\psi_r}^{-1}$$

hence

$$\vartheta_c^+ = S_r^+ V_{\psi_r}^{-1} \boldsymbol{\vartheta}_r = S^* V_{\psi_r}^{-1} \boldsymbol{\vartheta}_r = V_\psi S^\dagger V_{\psi_r}^{-1} \boldsymbol{\vartheta}_r$$

i.e.,  $\vartheta_c^+ = V_\psi \vartheta_{c+}$  (from Eqs. (34) and (40)). As a corollary (see Eq. (16)),

$$\begin{aligned} \|\vartheta_c^+\|_{E^+}^2 &= \langle \vartheta_c^+ | \vartheta_c^+ \rangle_{E^+} \\ &= \langle \vartheta_c^+ | V_\psi \vartheta_{c+} \rangle_{E^+} = (\vartheta_c^+ \cdot \vartheta_{c+})_E \end{aligned}$$

As  $\vartheta_c$  is the projection of  $\vartheta_c^+$  on  $E_c$  (see Fig. 4), we have (see Eqs. (40) and (18))

$$\begin{aligned} (\vartheta_c^+ \cdot \vartheta_{c+})_E &= (\vartheta_c \cdot \vartheta_{c+})_E \\ &= (\vartheta_c \cdot S^\dagger \boldsymbol{\vartheta}_{r+})_E \\ &= (S \vartheta_c \cdot \boldsymbol{\vartheta}_{r+})_F \end{aligned}$$

But, from Eq. (15),  $S \vartheta_c = \boldsymbol{\vartheta}_r$ . As a result,

$$\|\vartheta_c^+\|_{E^+}^2 = (\boldsymbol{\vartheta}_r \cdot \boldsymbol{\vartheta}_{r+})_F = \|\boldsymbol{\vartheta}_r\|_{F_{r+}}^2 \quad \square$$

### 4.2 Analysis of a typical situation

To illustrate our analysis, we now consider the case where the variance-covariance matrix of the observational data of type  $\psi$  is of the form

$$\mathbf{V}_\psi = \text{diag}(\eta(r_i, s_j) \sigma_\psi^2) \quad (41)$$

Clearly,  $\sigma_\psi^2$  is a ‘reference variance;’  $\eta(r, s)$  is a nonnegative weight function. The variance-covariance matrix of the SD data  $\boldsymbol{\psi} := S\boldsymbol{\psi}$  is then given by the relation

$$\mathbf{V}_\boldsymbol{\psi} = \text{diag}(\eta_j \sigma_\psi^2) \quad \eta_j := \eta(r_1, s_j) + \eta(r_2, s_j) \quad (42)$$

As clarified in Remark 4.2.1, the following results sheds a new light on the centralized observational approach of Shi and Han (1992). The dual approach of Lannes and Durand (2003) is also thereby enriched.

**Result 4.2.1.** Denoting by  $\vartheta_{r+}^j$  and  $\vartheta_r^j$  the components of  $\boldsymbol{\vartheta}_{r+}$  and  $\boldsymbol{\vartheta}_r$ , respectively, we have

$$\vartheta_{r+}^j = \frac{1}{\eta_j \sigma_\psi^2} (\vartheta_r^j - \delta_\vartheta) \quad (\vartheta_r^j \equiv \vartheta_{r+}^j)$$

where

$$\delta_\vartheta := \sum_{j=1}^n \mu_j \vartheta_r^j \quad \mu_j := \frac{1}{\sum_{j=1}^n \frac{1}{\eta_j}}$$

As a corollary,  $\vartheta_c^+ = \eta \sigma_\psi^2 \vartheta_{c+} = \eta \sigma_\psi^2 S^\dagger \boldsymbol{\vartheta}_{r+}$ .

**Proof.** By definition,  $\boldsymbol{\vartheta}_{r+} := V_{\psi_r}^{-1} \boldsymbol{\vartheta}_r$  (Eq. (34)). To identify the inverse of  $V_{\psi_r}$  on  $F_r$ , we solve the equation  $V_{\psi_r} \boldsymbol{\vartheta}' = \boldsymbol{\vartheta}_r$  in  $F_r$ . From Eq. (32),  $V_{\psi_r} \boldsymbol{\vartheta}'$  is equal to  $V_\psi \boldsymbol{\vartheta}'$  up to a vector of  $F_0$ . It then follows from Eq. (42) that the components of  $\boldsymbol{\vartheta}'$  are related to those of  $\boldsymbol{\vartheta}_r$  by the relation

$$\eta_j \sigma_\psi^2 \vartheta'^j = \vartheta_r^j - \delta$$



property, these authors should have described, explicitly, in the special case under consideration, the action of the pseudoinverse of operator  $S_d$  (the ‘closure operator’  $C$  of their formulation).

With regard to all these points, the more general results established in this section enrich both the dual algebraic formulation of differential GPS and the centralized observational approach.

**Remark 4.2.2.** In the special case under consideration (where the weights  $\eta(r_i, s_j)$  are all equal to unity), the identity expressed in the right-hand side of Eq. (44) can directly be derived from the traditional approach to differential GNSS. This can be shown as follows. For clarity, consider the case where  $k = 1$ . As is well known, the matrix elements of  $V_{\psi d}^{-1}$  are then given by the formula

$$\kappa_{j,j'} = \frac{1}{2\sigma_\psi^2} \times \frac{1}{n} \begin{cases} n-1 & \text{if } j' = j \\ -1 & \text{if } j' \neq j \end{cases} \quad j, j' \in \{2, \dots, n\}$$

Clearly, for any  $\vartheta$  in  $F_d$ , we have

$$\|\vartheta\|_{F_{d+}}^2 = (\vartheta \cdot V_{\psi d}^{-1} \vartheta)_F = \sum_{j=2}^n \vartheta_k^j (V_{\psi d}^{-1} \vartheta)^j$$

in which (for  $j = 2, \dots, n$ )

$$\begin{aligned} (V_{\psi d}^{-1} \vartheta)^j &= \frac{1}{2\sigma_\psi^2} \left( \vartheta_k^j - \frac{1}{n} \sum_{j=2}^n \vartheta_k^j \right) \\ &= \frac{1}{2\sigma_\psi^2} \left[ (\vartheta^j - \vartheta^k) - \frac{1}{n} \sum_{j=1}^n (\vartheta^j - \vartheta^k) \right] \\ &= \frac{1}{2\sigma_\psi^2} \left( \vartheta^j - \frac{1}{n} \sum_{j=1}^n \vartheta^j \right) \\ &= \frac{1}{2\sigma_\psi^2} \vartheta_0^j \end{aligned}$$

As a result,

$$\begin{aligned} \sum_{j=2}^n \vartheta_k^j (V_{\psi d}^{-1} \vartheta)^j &= \frac{1}{2\sigma_\psi^2} \sum_{j=2}^n (\vartheta^j - \vartheta^k) \vartheta_0^j \\ &= \frac{1}{2\sigma_\psi^2} \sum_{j=1}^n (\vartheta^j - \vartheta^k) \vartheta_0^j \\ &= \frac{1}{2\sigma_\psi^2} \sum_{j=1}^n (\vartheta_0^j - \vartheta_0^k) \vartheta_0^j \end{aligned}$$

Since  $\sum_{j=1}^n \vartheta_0^j = 0$ , it then follows that

$$\|\vartheta\|_{F_{d+}}^2 = \sum_{j=1}^n \frac{1}{2\sigma_\psi^2} |\vartheta_0^j|^2 \quad (\vartheta_0^j \equiv \vartheta_r^j)$$

## 5 Data assimilation in RD mode

In the statement of the global positioning problems, the position variable at epoch  $t$ ,  $\xi_t$ , appears via the linearization of the quantities  $\rho_t^j$  with respect to the position variable  $\xi_{2;t}$  of receiver  $r_2$ :  $\xi_{2;t} = \tilde{\xi}_{2;t} + \xi_t$ . Indeed, as

$$\rho_t^j = \rho_t(r_2, s_j) - \rho_t(r_1, s_j) \quad (45)$$

the linear expansion of  $\rho_t^j$  is of the form

$$\rho_t^j = \tilde{\rho}_t^j + (d_t^j \cdot \xi_t)_{\mathbb{R}^3} \quad (46)$$

Here,  $d_t^j$  is the unitary vector that characterizes the direction  $s_j \rightarrow r_2$  of the signal received at epoch  $t$ . Let  $J_t$  be the matrix whose elements of the  $j^{\text{th}}$  line are the three components of  $d_t^j$ . Denoting by  $J_t$  the corresponding operator, we thus have  $\rho_t = \tilde{\rho}_t + J_t \xi_t$ , hence

$$\rho_{t;r} = \tilde{\rho}_{t;r} + J_{t;r} \xi_t \quad (J_{t;r} := Q_r J_t) \quad (47)$$

In single-frequency mode, the state variable at epoch  $t$ , the local variable  $x_t$ , is the column matrix

$$x_t := (\alpha, \xi_t)^T \quad (48)$$

with  $\alpha \equiv \mathbf{a}_r$  in  $F_r$ . The global variable for the epochs  $t_1, t_2, \dots, t_n$  is then of the form

$$X := (\alpha, \xi_1, \xi_2, \dots, \xi_n)^T \quad (49)$$

where  $\xi_n \equiv \xi_{t_n}$ . Clearly, the ‘float ambiguity’  $\alpha$  does not depend on  $t$ . Let  $y_t$  be the RD data vector (at epoch  $t$ ) modified by the terms induced by the linearization:

$$y_t := \begin{pmatrix} \mathbf{p}_{t;r} - \tilde{\rho}_{t;r} \\ \phi_{t;r} - \tilde{\rho}_{t;r} \end{pmatrix} \quad (50)$$

We then have

$$y_t = A_t x_t + \text{error terms} \quad (51)$$

where

$$A_t := \begin{pmatrix} 0 & J_{t;r} \\ \lambda I_\alpha & J_{t;r} \end{pmatrix} \quad (52)$$

The problem is to be solved in the least-square sense at the global level. We then introduce the column matrix

$$Y = (y_1, y_2, \dots, y_n)^T \quad (53)$$

where  $y_n \equiv y_{t_n}$ . Clearly,

$$Y = AX + \text{error terms} \quad (54)$$



where the global operator  $A$  is then of the form:

$$A := \begin{pmatrix} \cdot & J_{1;r} & \cdot & \cdots & \cdot \\ \lambda I_\alpha & J_{1;r} & \cdot & \cdots & \cdot \\ \hline \cdot & \cdot & J_{2;r} & \cdots & \cdot \\ \lambda I_\alpha & \cdot & J_{2;r} & \cdots & \cdot \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \cdot & \cdot & \cdot & \cdots & J_{n;r} \\ \lambda I_\alpha & \cdot & \cdot & \cdots & J_{n;r} \end{pmatrix} \quad (55)$$

## 5.1 Recursive least-square filtering

The solution  $\hat{x} \equiv (\hat{\alpha}, \hat{\xi})^\top$  is obtained through recursive least-squares (RLS) filtering (e.g., Björck 1996). The iteration at epoch  $t_n$  is then of the form

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n v_n \quad (56)$$

in which

$$v_n = y_n - A_n \hat{x}_{n|n-1} \quad (57)$$

where

$$\hat{x}_{n|n} := (\hat{\alpha}_n, \hat{\xi}_n) \quad \hat{x}_{n|n-1} := (\hat{\alpha}_{n-1}, 0) \quad (58)$$

Clearly,  $K_n$  is the RLS filter at epoch  $t_n$ ;  $v_n$  is the ‘predicted residue’ for the same epoch. The float solution  $\hat{\alpha}$  is thus refined together with its variance-covariance matrix  $V_{\hat{\alpha}}$ .

## 5.2 Ambiguity resolution

At each epoch, one then searches for the point  $\check{\alpha}$  of  $L_r$  closest to  $\hat{\alpha}$  the distance being that induced by the quadratic form on  $F_r$ :  $f(\vartheta) := (\vartheta \cdot V_{\hat{\alpha}}^{-1} \vartheta)_F$ . Clearly,

$$\check{\alpha} = \operatorname{argmin}_{\alpha \in L_r} \|\alpha - \hat{\alpha}\|_{V_{\hat{\alpha}}^{-1}} \quad (59)$$

As specified in Agrell et al. (2002), this nearest-lattice-point problem is solved via the LLL algorithm, an algorithm devised by Lenstra, Lenstra and Lovász in 1982. To state this problem in technical terms, we then have to choose a reference basis of  $L_r$ . The most natural choice corresponds to a basis such as  $\mathbf{b}_{rd}$  (see Sect. 3). The reference index  $k$  can then be chosen arbitrarily, for example, the first one or the last one of the current list of satellites. In this basis, the components of  $\alpha$  are the components of  $\alpha_d := D\alpha$  in basis  $\mathbf{b}_d$  (see the analysis developed in Sect. 3). Likewise, the components of  $\hat{\alpha}$  in basis  $\mathbf{b}_{rd}$  are the components of  $\hat{\alpha}_d := D\hat{\alpha}$  in basis  $\mathbf{b}_d$ . Furthermore, the variance-covariance matrix of  $\hat{\alpha}$  expressed in

basis  $\mathbf{b}_{rd}$  is equal to  $DV_{\hat{\alpha}}D^*$ , i.e.,  $V_{D\hat{\alpha}} = V_{\hat{\alpha}_d}$ . Solving the RD ambiguity problem in  $L_r$  therefore amounts to solving the integer-ambiguity problem of the DD approach in  $L_d$  (e.g., Teunissen 1995):

$$\check{\alpha}_d = \operatorname{argmin}_{\alpha_d \in L_d} \|\alpha_d - \hat{\alpha}_d\|_{V_{\hat{\alpha}_d}^{-1}} \quad (60)$$

Clearly  $\check{\alpha} = R\check{\alpha}_d$ . The explicit statement (in a trivial basis) of the ambiguity problem in question is therefore that of the DD approach. This does not mean, of course, that the DD approach is the best suited for stating all the problems of the data assimilation process (see Sect. 6 in particular).

At the technical level, we therefore proceed as follows. We first set  $k = 1$  for example. Starting from the float RD ambiguity vector  $\hat{\alpha}$  and its variance-covariance matrix  $V_{\hat{\alpha}}$ , we then compute the DD float ambiguity vector  $\hat{\alpha}_d = D\hat{\alpha}$  and its variance-covariance matrix  $V_{\hat{\alpha}_d} = DV_{\hat{\alpha}}D^*$ . In single-frequency mode, the  $n - 1$  components of  $\hat{\alpha}_d$  are therefore explicitly given by the relations

$$\hat{\alpha}_k^j = \hat{\alpha}^j - \hat{\alpha}^k \quad (j = 2, \dots, n) \quad (61)$$

Similar operations on the columns and lines of  $V_{\hat{\alpha}}$  provide the  $(n - 1)^2$  matrix elements of  $V_{\hat{\alpha}_d}$ . Solving the DD ambiguity problem (60) then provides the DD ambiguity vector  $\check{\alpha}_d$ . In single-frequency mode, the components of the RD ambiguity vector  $\check{\alpha} = R\check{\alpha}_d$  are then explicitly given by the relations

$$\check{\alpha}^j = \check{\alpha}_k^j - \check{\alpha}_k^0 \quad \check{\alpha}_k^0 := \frac{1}{n} \sum_{j \neq k} \check{\alpha}_k^j \quad (62)$$

Clearly, we thus have passed from the RD framework to the DD framework (and then vice-versa) only to solve the technical problem in question.

## 5.3 Fixed ambiguities

When in the data assimilation process,  $\check{\alpha}$  becomes consistent with the model (up to the noise), the ambiguities are said to be fixed. The positions  $\hat{\xi}_n$  are then refined via ‘FLS’ filtering: LS processes in which the ambiguities are fixed at these values. Processing the same undifferential data either in RD or DD mode of course provides the same positions.

To prevent that biases on the SD data propagate undetected into the ambiguity solution and the positioning results, particular methods have been developed. These DIA methods ‘Detect’ these model errors, ‘Identify’ them, and ‘Adapt’ the results consequently (e.g., Teunissen 1990, Hewitson et al. 2004). We now show how the related RLS and FLS procedures can benefit from the RD approach.

## 6 DIA methods in RD mode

In RLS mode (for example), the principle of the RD version of these methods is based on the analysis of the ‘actual residue’

$$w_n := y_n - A_n \widehat{x}_{n|n} = H_n v_n \quad (63)$$

where, from Eqs. (56) and (57),

$$H_n := I - A_n K_n \quad (64)$$

Here,  $I$  is the identity operator. Omitting subscript  $n$ , and denoting by  $w_p$  and  $w_\phi$  the code and phase components of  $w$  (respectively), we have from Result 4.2.2 and 4.4.1

$$\|w\|^2 := \|w_p\|_{F_{p;r+}}^2 + \|w_\phi\|_{F_{\phi;r+}}^2 \quad (65)$$

where (for  $\psi = p$  or  $\phi$ )

$$\|w_\psi\|_{F_{\psi;r+}}^2 = \sum_{j_\psi=1}^n c_{j_\psi} \quad (66)$$

with

$$c_{j_\psi} := \frac{1}{\eta_j \sigma_\psi^2} (w_\psi^j - \delta_{w_\psi})^2 \quad \delta_{w_\psi} := \sum_{j=1}^n \mu_j w_\psi^j \quad (67)$$

When  $\|w\|^2$  is too large, above some threshold defined by statistical criteria (see Sect. 6.1), we then search to identify a global SD bias of the form

$$\beta = \left( \sum_{j_p \in \Omega_p} \beta_{j_p} e_{j_p}, \sum_{j_\phi \in \Omega_\phi} \beta_{j_\phi} e_{j_\phi} \right) \quad (68)$$

The ‘outlier sets’  $\Omega_p$  and  $\Omega_\phi$  are some ‘small subsets’ of  $\{1, \dots, n\}$ . The corresponding SD model is the following (see Sect. 1):

$$\rho^j + c[\text{dt}(r_2) - \text{dt}(r_1)] + \epsilon^j = \begin{cases} p^j - \beta_{j_p} & \text{if } j \in \Omega_p \\ p^j & \text{otherwise} \end{cases}$$

for the code, and likewise for the phase (see Eq. (5)). The problem is to identify  $\Omega_p$  and  $\Omega_\phi$  while getting least-squares estimates of the corresponding biases  $\beta_{j_p}$  and  $\beta_{j_\phi}$ . The guiding idea is to consider the contribution of these biases to  $w$ .

As  $w = H \delta v = H \delta y$  (see Eqs. (63) and (57)), we must first see what is the contribution of these biases to  $y$ . At this level, the correction terms induced by  $e_{j_p}$  and  $e_{j_\phi}$  are denoted by  $z_{j_p}$  and  $z_{j_\phi}$ :

$$y \stackrel{\text{set}}{=} y - z_{j_\psi} \quad \begin{cases} z_{j_p} := (e_{r_{j_p}}, 0) \\ z_{j_\phi} := (0, e_{r_{j_\phi}}) \end{cases} \quad (69)$$

Clearly, a notation such as  $a \stackrel{\text{set}}{=} a + b$  means ‘ $a$  is set equal to the current value of  $a + b$ .’ The components of the

basis vectors  $e_{r_j}$  are explicitly defined by the relations (see Sect. 3)

$$\begin{cases} e_{r_j}^{j'} = -1/n & (\text{for } j' \neq j); \\ e_{r_j}^j = 1 - 1/n \end{cases}$$

The variations of  $w$  induced by  $e_{j_p}$  and  $e_{j_\phi}$  are therefore characterized by the quantities  $f_{j_p}$  and  $f_{j_\phi}$  defined below:

$$w \stackrel{\text{set}}{=} w - H z_{j_\psi} \quad \begin{cases} f_{j_p} := H z_{j_p} \\ f_{j_\phi} := H z_{j_\phi} \end{cases} \quad (70)$$

As a result, the variation of  $w$  induced by the global bias  $\beta$  is characterized by the vector

$$M\beta := \sum_{j_p \in \Omega_p} \beta_{j_p} f_{j_p} + \sum_{j_\phi \in \Omega_\phi} \beta_{j_\phi} f_{j_\phi} \quad (71)$$

We are then led to solve, in the least-square sense, the equation  $M\beta \stackrel{=}{=} w$ , in which the column vectors of  $M$ , the  $f_{j_p}$ ’s and  $f_{j_\phi}$ ’s, have to be thoroughly selected. As clarified in Sect. 6.1, this operation is performed via a particular Gram-Schmidt orthogonalization process which is interrupted as soon as the corrected data are consistent with the model. As expected, Equations (65), (66) and (67) play a key role in the identification of the global outlier set  $\Omega := \Omega_p \cup \Omega_\phi$ .

### 6.1 Implementation

In the procedure described in this section (see the flow diagram shown in Fig. 6),  $\theta$  is the level of significance or the probability of false alarm of the local overall model (LOM) test;  $\theta_0$  is that of the outlier test.

#### 1. Entrance LOM test

Compute  $T_{\text{LOM}} := \|w\|^2 / \mathbf{m}$  where  $\mathbf{m} = 2(n-1) - 3$  is the redundancy (in the single-frequency case) at the current epoch. Let  $t_{\text{LOM}} := F_\theta(\mathbf{m}, \infty, 0)$  now be the upper  $\theta$  probability point of the central  $F$ -distribution with  $\mathbf{m}, \infty$  degrees of freedom. If  $T_{\text{LOM}} < t_{\text{LOM}}$ , terminate the process (go to step 4); otherwise, set  $\tau = 1$  (the recursive index) and  $\Omega = \Pi = \emptyset$  (the empty set); the meaning of the auxiliary set  $\Pi$  is defined in step 2.2 (as soon as it begins to be built).

#### 2. Recursive identification of the outliers

##### 2.1. Current set of potential outliers

For all the  $j_\psi \notin \Omega$ , compute the components  $c_{j_\psi}$  of  $\|w\|^2$  and their maximal value:

$$c_{\max} := \max_{j_\psi \notin \Omega} c_{j_\psi}$$

Then, given some nonnegative constant  $\kappa \leq 1$ , form the current set of potential outliers (see Fig. 5):

$$\Pi_\tau := \{j_\psi \notin \Omega : c_{j_\psi} \geq \kappa c_{\max}\}$$

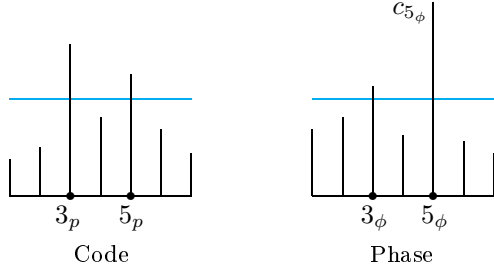


Fig. 5 *Notion of potential outliers* (in RD single-frequency mode). For the components  $c_{j_\psi}$  shown here, and for  $\kappa = 0.5$  (with  $n = 7$  and  $\Omega = \emptyset$ ), four potential outliers are identified:  $3_p$ ,  $5_p$ ,  $3_\phi$  and  $5_\phi$ . Here, the phase outlier  $5_\phi$  is likely to be the dominant potential outlier (see step 2.3 and Sect. 6.2).

## 2.2. For each potential outlier $j_\psi \in \Pi_\tau$

Perform the following successive operations:

- a) When  $j_\psi \notin \Pi$ , compute (see the context of Eqs. (69) and (70))

$$f_{j_\psi} := H \cdot \begin{cases} (\mathbf{e}_{r_{j_p}}, 0) & \text{if } \psi = p \\ (0, \mathbf{e}_{r_{j_\phi}}) & \text{if } \psi = \phi \end{cases}$$

Then, set

$$g_{j_\psi} := f_{j_\psi} \quad \Pi \stackrel{\text{set}}{=} \begin{cases} \{j_\psi\} & \text{if } \Pi = \emptyset \\ \Pi \cup \{j_\psi\} & \text{otherwise} \end{cases}$$

By construction,  $\Pi$  is the set of potential outliers  $j_\psi$  for which  $f_{j_\psi}$  has already been computed.

- b) If  $\tau = 1$  go to step 2.2c. Otherwise, at this level,  $\{g_q^\circ\}_{q < \tau}$  is an orthonormal set. (This set is built, progressively, via step 2.4.) Then, for each integer  $q < \tau$ , consider the inner product defined as follows (see Eq. (34) and Result 4.2.1):

$$\begin{aligned} \varsigma_{q,j_\psi} &:= \langle g_q^\circ | g_{j_\psi} \rangle \\ &:= \sum_{\psi'=p,\phi} \langle \mathbf{g}_{q;\psi'}^\circ | \mathbf{g}_{j_\psi;\psi'} \rangle_{F_{\psi';\tau+}} \end{aligned}$$

This sum includes two terms. Depending on what  $\psi'$  refers to ( $p$  or  $\phi$ ),  $\mathbf{g}_{q;\psi'}^\circ$  denotes the code or phase component of  $g_q^\circ$ , and likewise for  $\mathbf{g}_{j_\psi;\psi'}$ . If  $\varsigma_{q,j_\psi}$  has not been computed yet, compute it, store it in memory, and perform the Gram-Schmidt orthogonalization operation

$$g_{j_\psi} \stackrel{\text{set}}{=} g_{j_\psi} - \varsigma_{q,j_\psi} g_q^\circ$$

By construction,  $\varsigma_{q,j_\psi} = \langle g_q^\circ | f_{j_\psi} \rangle$ . Clearly, at the end of all these operations,  $g_{j_\psi}$  is orthogonal to  $g_q^\circ$  for any  $q < \tau$ .

- c) Consider the projection of  $w$  on the one-dimensional space generated by  $g_{j_\psi}$ , i.e.,  $\langle h_{j_\psi} | w \rangle h_{j_\psi}$  where

$h_{j_\psi} := g_{j_\psi} / \|g_{j_\psi}\|$ . The norm of this projection is equal to  $|\langle h_{j_\psi} | w \rangle|$ , the absolute value of the quantity

$$\gamma_{j_\psi} := \langle g_{j_\psi} | w \rangle / \varrho_{j_\psi} \quad \varrho_{j_\psi} := \|g_{j_\psi}\|$$

Explicitly,

$$\begin{aligned} \langle g_{j_\psi} | w \rangle &:= \sum_{\psi'=p,\phi} \langle \mathbf{g}_{j_\psi;\psi'} | \mathbf{w}_{\psi'} \rangle_{F_{\psi';\tau+}} \\ \|g_{j_\psi}\|^2 &:= \sum_{\psi'=p,\phi} \|\mathbf{g}_{j_\psi;\psi'}\|_{F_{\psi';\tau+}}^2 \end{aligned}$$

## 2.3. Dominant potential outlier

By definition, the dominant potential outlier  $\bar{j}_\psi$  is the potential outlier for which  $|\gamma_{j_\psi}|$  is maximal:

$$\bar{j}_\psi := \arg \max_{j_\psi \in \Pi_\tau} |\gamma_{j_\psi}|$$

## 2.4. Outlier test

Let  $\chi_0$  be the upper  $\theta_0/2$  probability point of the central normal distribution:

$$\chi_0 := N_{\theta_0/2}(0, 1)$$

- a) If  $|\gamma_{\bar{j}_\psi}| > \chi_0$  with  $\mathbf{m} > 0$ , the dominant potential outlier is then regarded as an effective outlier:

$$\omega_\tau := \bar{j}_\psi \quad \Omega \stackrel{\text{set}}{=} \begin{cases} \{\omega_\tau\} & \text{if } \tau = 1 \\ \Omega \cup \{\omega_\tau\} & \text{if } \tau > 1 \end{cases}$$

$$\gamma_\tau^\circ := \gamma_{\omega_\tau} \quad g_\tau^\circ := g_{\omega_\tau} / \varrho_{\omega_\tau}$$

Superscript  $\circ$  stands for omega (and outlier). At this level,  $\Omega$  is the current set of identified outliers:

$$\Omega = \{\omega_q\}_{q=1}^\tau$$

By construction,  $\{g_q^\circ\}_{q=1}^\tau$  is an orthonormal basis of the current range of  $M$ ;  $\sum_{q=1}^\tau \gamma_q^\circ g_q^\circ$  is the projection of  $w$  on this space. With regard to Eq. (71), we then set

$$\beta_\tau^\circ := \beta_{\omega_\tau} \quad f_\tau^\circ := f_{\omega_\tau}$$

- b) When the dominant potential outlier is not identified as a real outlier, we consider the following two situations:

*Case 1:*  $|\gamma_{\bar{j}_\psi}| < \chi_0$  with  $T_{\text{LOM}} > 5 t_{\text{LOM}}$  (for example). We then reinitialize the RLS process.

*Case 2:*  $|\gamma_{\bar{j}_\psi}| < \chi_0$  with  $T_{\text{LOM}} < 5 t_{\text{LOM}}$  and  $\tau > 1$ , or  $|\gamma_{\bar{j}_\psi}| > \chi_0$  with  $\mathbf{m} = 0$ . We then go to step 3.

## 2.5. Components of $g_\tau^\circ$ in the basis of the $f_q^\circ$ 's

These components are denoted by  $u_{q,\tau}$ :

$$g_\tau^\circ = \sum_{q=1}^\tau u_{q,\tau} f_q^\circ$$

They are computed via the QR Gram-Schmidt formulas (see e.g., Björck 1996)

$$u_{q,\tau} = \begin{cases} -\frac{1}{\varrho_{\omega_\tau}} \sum_{q \leq q' < \tau} u_{q,q'} \varsigma_{q',\omega_\tau} & \text{if } q < \tau \\ \frac{1}{\varrho_{\omega_\tau}} & \text{if } q = \tau \end{cases}$$

for  $1 \leq q \leq \tau$ . Clearly, the  $u_{q,\tau}$ 's are the entries of the  $\tau^{\text{th}}$  column of an upper triangular matrix  $U$ .

### 2.6. SD biases

According to Eq. (71), the SD biases  $\beta_q^\circ$  are the components of  $\sum_{q=1}^{\tau} \gamma_q^\circ g_q^\circ$  in the basis of the  $f_q^\circ$ 's:

$$\sum_{q=1}^{\tau} \gamma_q^\circ g_q^\circ = \sum_{q=1}^{\tau} \beta_q^\circ f_q^\circ$$

Denoting by  $[\gamma^\circ]$  the column matrix with entries  $\gamma_q^\circ$  (from  $q = 1$  to  $\tau$ ), and likewise for  $[\beta^\circ]$ , we have

$$[\beta^\circ] = U[\gamma^\circ]$$

The SD biases are therefore to be updated as follows:

$$\beta_q^{\text{set}} \equiv \begin{cases} \beta_q^\circ + u_{q,\tau} \gamma_\tau^\circ & \text{if } q < \tau \\ u_{\tau,\tau} \gamma_\tau^\circ & \text{if } q = \tau \end{cases} \quad (\text{for } 1 \leq q \leq \tau)$$

### 2.7. Update $w$ and $\|w\|^2$ :

$$w^{\text{set}} = w - \gamma_\tau^\circ g_\tau^\circ \quad \|w\|^2 \stackrel{\text{set}}{=} \|w\|^2 - |\gamma_\tau^\circ|^2$$

### 2.8. Update redundancy $m$ and the LOM quantities:

$$m \stackrel{\text{set}}{=} m - 1 \quad \left| \begin{array}{l} t_{\text{LOM}} \stackrel{\text{set}}{=} F_\theta(m, \infty, 0) \\ T_{\text{LOM}} \stackrel{\text{set}}{=} \|w\|^2 / m \end{array} \right.$$

### 2.9. Inner LOM test

If  $T_{\text{LOM}} > t_{\text{LOM}}$ , update recursive index:  $\tau \stackrel{\text{set}}{=} \tau + 1$ . Then, go to step 2.

### 3. Local adaptation

Let  $\mathbf{K}_\Omega$  be the matrix gathering the columns of  $K$  corresponding to the successive identified outliers  $\omega_1, \dots, \omega_\tau$ . The adaptation formula of the local state vector is then (from Eqs. (56) and (57))

$$\hat{x} \stackrel{\text{set}}{=} \hat{x} - \mathbf{K}_\Omega [\beta^\circ]$$

As  $[\beta^\circ] = U[\gamma^\circ]$ , the adaptation of the variance-covariance matrix of  $\hat{x}$  is therefore given by the formula

$$\mathbf{V}_{\hat{x}} \stackrel{\text{set}}{=} \mathbf{V}_{\hat{x}} + [\mathbf{K}_\Omega U][\mathbf{K}_\Omega U]^T$$

Indeed, as  $\{g_q^\circ\}_{q=1}^{\tau}$  is an orthonormal set, the variance-covariance matrix of  $[\gamma^\circ]$  is the identity.

### 4. End

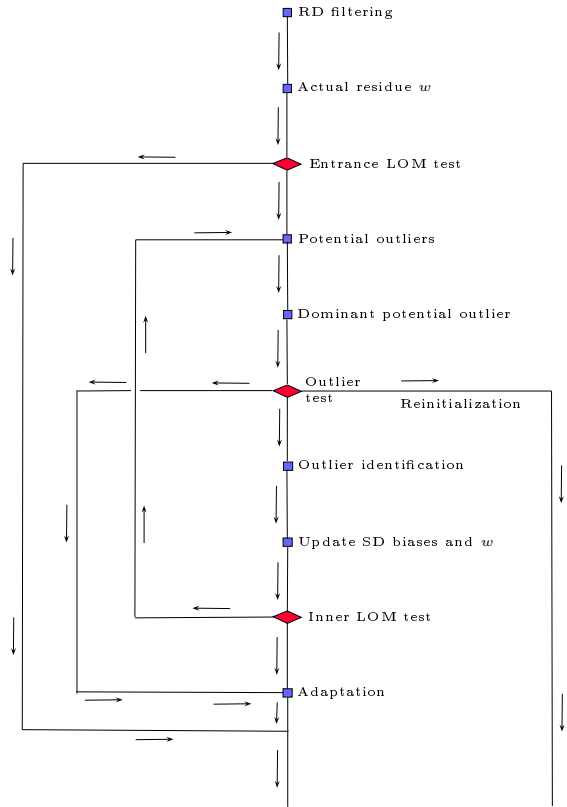


Fig. 6 Flow diagram of the DIA procedure in RD mode. At each step of the identification process, the 'residual actual residue'  $w$  is analyzed on the grounds of Eq. (72) or of its generalization (see Fig. 5, Eqs. (65), (66) and (67)). This allows the potential outliers to be selected. The outliers can thus be identified, in a recursive manner, via a particular orthogonalization Gram-Schmidt process. This QR Gram-Schmidt process also provides the SD biases, and thereby the cycle slips if any (see text).

In order to detect a model error of the same size with the same probability  $\varpi_0$  by using both LOM and outlier tests, it is required that, for both tests, the same values for the non-centrality parameter  $\zeta_0$  be chosen.

To determine the testing parameters, one therefore proceeds as follows. One first makes a choice for  $\theta_0$  and  $\varpi_0$ :

$$\theta_0 = 0.001 \quad \varpi_0 = 0.80 \quad (\text{for example})$$

The non-centrality parameter  $\zeta_0$  of both tests is computed from these values. One then obtains the critical value  $F_\theta(m, \infty, 0)$  of the LOM test, and thereby  $\theta$ .

## 6.2 Examples

The RD approach was validated in the framework of a European action entitled HPLE.<sup>1</sup> Real GPS data were

<sup>1</sup>The HPLE (High Precision Local Element) project was co-funded by the European GNSS Supervisory Authority with funding from the Sixth Framework Programme of the European Community for research and technological development, European Union's

thus processed in the dual and single-frequency modes. For all these data sets, the DIA procedure was conducted with  $\eta_j = 2$  for all  $j$ .

In the single-frequency mode, Equation (65) then reduces to

$$\|w\|^2 = \frac{1}{2\sigma_p^2} \sum_{j=1}^n |w_p^j|^2 + \frac{1}{2\sigma_\phi^2} \sum_{j=1}^n |w_\phi^j|^2 \quad (72)$$

As a general rule, at each step of the recursive identification process, the  $j_p$  or  $j_\phi$  to be selected, the dominant potential outlier, then corresponds to the maximal value of  $|w_p^j|/\sigma_p$  and  $|w_\phi^j|/\sigma_\phi$  for  $j = 1, \dots, n$  (see Fig. 5). As illustrated in the following examples, this is also the case in dual-frequency mode.

We now present related results concerning a set of GPS data provided by the French DGA for testing: 4907 epochs at 1 Hz in dual-frequency mode (L1-C/A, L2-P) with many appearances and disappearances of satellites. Over this time series, depending on the epochs, their number was 7, 8 or 9.

The reference and user receivers were static. The relative Cartesian coordinates of the user receiver were of the order of  $-303$  m,  $121$  m and  $238$  m.

The data set in question was reinitialized at the following epochs: 1094, 1301, 3010 and 4689. The ambiguities were then fixed in one or two seconds: the position of the user receiver was thus retrieved, up to one centimeter, except for epochs 1, 1094, 1301–1302, 3010–3011 and 4689–4690.

Table 1 *Dual-frequency DD ambiguities  $\check{a}_{f_\nu; k}^j$  for  $j = 1, \dots, n$  with  $n = 9$  and  $k = 1$ . At the epoch under consideration, these ambiguities were fixed; for their RD transcription, see text (and Sect. 5.2, in particular).*

$j; f_\nu$	$f_1$	$f_2$
1	0	0
2	34 868	257 496
3	625 263	−196 104
4	−2 502 896	−1 419 324
5	12 155 323	9 705 967
6	−2 593 167	−1 303 294
7	5 973 773	4 346 092
8	9 056 740	7 252 801
9	−9 386 838	−7 332 507

To illustrate the dual-frequency version of the approach presented in this paper, we now concentrate on the pro-

chief instrument for funding research. The European GNSS Supervisory Authority is the EC agency in charge of the implementation of Galileo, Europe's future satellite navigation system.

cess at epoch 4745. Nine satellites were then available: satellites 1, 5, 7, 8, 9, 21, 23, 26, 30 ( $j = 1, \dots, 9$ ).

At that epoch, the ambiguity vector ( $\check{a}_{f_1}, \check{a}_{f_2}$ ) was fixed. The DD transcription of its components is displayed in Table 1 for  $k = 1$ .

For  $j = 1, \dots, n$ , and for each frequency, the corresponding RD ambiguities are then given by Eq. (62) with  $k = 1$ ,  $\check{\alpha} \stackrel{\text{set}}{=} \check{a}_{f_1}$  and  $\check{\alpha} \stackrel{\text{set}}{=} \check{a}_{f_2}$ . It was of course verified that the estimated relative coordinates of the user receiver were exactly the same in both approaches:  $-303.39$  m,  $120.92$  m and  $238.49$  m (the correct values up to one centimeter).

The DIA procedure implemented in that case was the FLS dual-frequency version of that presented in Section 6.1. The following two situations were then considered:

- A. Without any cycle slip (the real data)
- B. With the following added cycle slips (just to show the efficiency of the method):
  - 2 cycles in the reception of the  $f_1$ -signal coming from satellite 5 ( $j = 2$ );
  - $-1$  cycle in the reception of the  $f_2$ -signal coming from satellite 8 ( $j = 4$ );

#### A. Detection and identification without any cycle slip

In that case, the entrance value of  $T_{\text{LOM}}$  (9.13) was greater than the corresponding value of  $t_{\text{LOM}}$  (1.26). The outliers were then identified in the following order:

Outlier	$T_{\text{LOM}}$
$(f_2; 1_p)$	5.40
$(f_1; 4_\phi)$	2.63
$(f_1; 1_p)$	1.63
$(f_1; 8_p)$	1.06

The value in the right-hand side column is the corresponding reduced value of  $T_{\text{LOM}}$ . The last value of  $T_{\text{LOM}}$  (1.06) is smaller than the corresponding value of  $t_{\text{LOM}}$  (1.30). The biases thus found are displayed in Table 2.

#### B. Detection and identification with cycle slips

The entrance value of  $T_{\text{LOM}}$  was then very large (335.09), much greater than the corresponding value of  $t_{\text{LOM}}$  (1.26). The outliers were then identified in the following order:

Outlier	$T_{\text{LOM}}$
$(f_1; 2_\phi)$	108.81
$(f_2; 4_\phi)$	9.42
$(f_2; 1_p)$	5.41
$(f_1; 4_\phi)$	2.82
$(f_1; 1_p)$	1.74
$(f_1; 8_p)$	1.13

The last value of  $T_{LOM}$  (1.13) is smaller than the corresponding value of  $t_{LOM}$  (1.34). The biases thus found are displayed in Table 3.

All over the time series under consideration, the results were the same with  $\kappa = 1$  or  $\kappa = 0$  (see step 2.1 and Fig. 5 in Sect. 6.1). It is important to note that the choice  $\kappa = 0$ , which induces some CPU overhead (see step 2.2 in Sect. 6.1), implicitly corresponds the DD implementation of the DIA procedure by the Teunissen group at the Technical University of Delft (TUD). In Kalman mode,  $\kappa$  should likely be set equal to a smaller value (say 0.5 as in Fig. 5). This point remains to be investigated.

Table 2 Identification of a set of SD biases. The biases  $\beta_{f_\nu; j_\psi}$  are expressed in meters. At the epoch under consideration, nine satellites were available:  $n = 9$  (see text).

Frequency $f_1$				
$j_\psi$	1	2	4	8
$p$	-4.806			-3.304
$\phi$			0.043	

Frequency $f_2$				
$j_\psi$	1	2	4	8
$p$	-8.755			

Table 3 Identification of a set of SD biases including cycle slips. The situation is the same as that defined in Table 2, but with added cycle slips. The latter are correctly retrieved:  $\beta_{f_1; 2_\phi} \simeq 2\lambda_1$ ,  $\beta_{f_2; 4_\phi} \simeq -\lambda_2$ . Note that the identification order is, first, that induced by the cycle slips, and then, that displayed in case A (see text and Table 2).

Frequency $f_1$				
$j_\psi$	1	2	4	8
$p$	-4.806			-3.304
$\phi$		<b>0.381</b>	0.043	

Frequency $f_2$				
$j_\psi$	1	2	4	8
$p$	-8.755			
$\phi$			<b>-0.248</b>	

## 7 Concluding comments

The vertices of a GNSS graph are the receivers and the satellites of the GNSS device (see Lannes and Durand 2003). Its edges are the receiver-satellite pairs. The original observations are defined on these edges (see Sect. 2.1). As these observations are defined up to vectors in the nuisance delay space (see Sect. 2.1.1 and Fig. 4), the orthogonal complement of this space in the observational

data space plays a key part in the data assimilation procedures. In particular, brought back to this orthogonal complement, the residual quantities to be considered in the DIA procedures take their values on the edges of the graph.

To stress what is essential, the analysis presented in this paper was restricted to the special case where the GNSS graph includes only two receivers. On the two edges involved in the definition of a single difference, the double centralized observations of Shi and Han (1992) are then opposite. As clarified in Sect. 1.1.5, the information contained in these observations is then a simple antisymmetric transcription of that contained in the RD data.

The DD and RD approaches prove to be equivalent. More precisely, as specified in Sect. 3, the choice of the reference satellite induces that of a reference basis of the RD data space. The components of a RD vector in this basis are the corresponding DD's. Solving the problem in DD mode therefore amounts to solving it in this basis. At any stage of the data assimilation procedure, one may therefore pass from the RD mode to the DD mode, and vice-versa. In particular, solving the rational-ambiguity problem of the RD mode amounts to solving a nearest-lattice-point problem of DD type (see Sect. 5.2).

In RD mode, all the satellites are handled in the same manner. As a result, the numerical codes of the RD data assimilation processes are more readable than those of their DD versions. For example, in RD mode, the disappearance of the reference satellite of the DD approach is handled like that of any satellite.

This said, the main interest of the RD approach lies in the properties revealed by the corresponding 'dual analysis' (see Sect. 4). These properties, which are masked in the DD approach (see Remark 4.2.2), shed a new light on the CO approach of Shi and Han (1992). In particular, Result 4.2.2 can be exploited in the DIA procedures. From this point of view, Equation (43) is very significant. The notion of potential outlier derives from its corollary, the Euclidean quadratic decomposition (44); see Fig. 5. These properties also complete the contribution of Lannes and Durand (2003). All these aspects are analyzed and commented in Remark 4.2.1. As a result, all these equivalent approaches can benefit from each other.

The DIA procedure described in Section 6 follows the main guidelines of the DIA method of the TUD group (see, e.g., Fig. 6 in Teunissen 1990). In particular, the recursive detection process is based on a Gram-Schmidt orthogonalization procedure. The main new points derive from the notion of potential outliers. The orthogonalization procedure was implemented accordingly. The efficiency of the DIA method is thus improved. This particular implementation also benefits from the QR Gram-Schmidt step 2.5 of Sect. 6.1. The QR approach of the PhD dissertation of Tiberius (1998) can thereby be nicely completed. As specified in step 2.6 of Sect. 6.1, the SD biases can thus be recursively refined. The identification of

cycle slips, in particular, is performed in this way (see Table 3).

The GNSS graph may include more than two receivers; some receiver-satellite edges may also be missing. In this general situation, that of GNSS networks with missing data, it is important to benefit from all the redundancy of the data. The ‘identifiable biases’ must then be identified on the edges (or pairs of edges) where they appear. To solve the related problems in an efficient manner, the DD and CO approaches have to be conjugated and generalized in the ‘projected observational framework’ of Fig. 4. The related developments will be presented in a forthcoming paper (Lannes 2008).

## References

- Agrell E., Eriksson T., Vardy A. and Zeger K. (2002) *Closest point search in lattices*. IEEE Trans. Inform. Theory. 48: 2201–2214.
- Björck A. (1996) *Numerical methods for least-squares problems*. SIAM.
- Hewitson S., Lee H.K. and Wang J. (2004) *Localizability analysis for GPS/Galileo receiver autonomous integrity monitoring*. The Journal of Navigation, Royal Institute of Navigation 57: 245–259.
- Lannes A. and Durand S. (2003) *Dual algebraic formulation of differential GPS*. J. Geod. 77: 22–29.
- Lannes A. (2008) *GNSS networks with missing data: identifiable biases and potential outliers*. Proc. ENC GNSS-2008. Toulouse, France.
- Shi P. H. and Han S. (1992) *Centralized undifferential method for GPS network adjustment*. Australian Journal of Geodesy, Photogrammetry and Surveying. 57: 89–100.
- Strang G. and Borre K. (1997) *Linear algebra, geodesy, and GPS*, Wellesley-Cambridge Press, Massachusetts.
- Teunissen P.J.G. (1990) *An integrity and quality control procedure for use in multi sensor integration*. Proc. ION GPS-90. Colorado Springs, Colorado USA: 513–522
- Teunissen P.J.G. (1995) *The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation*. J. Geod. 70: 65–82.
- Tiberius C.C.J.M. (1998) *Recursive data processing for kinematic GPS surveying*. Publications on Geodesy. New series: ISSN 0165 1706, Number 45. Netherlands Geodetic Commission, Delft.
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